4) \( f \) continuous on \([a, b]\). Thus, by the Max-Min Theorem there exists \( r, p \in [a, b] \) such that:
\[
f(r) \leq f(x) \leq f(p) \quad \text{for all } x \in [a, b].
\] (1)
Let \( R = \{ f(x) : x \in [a, b] \} \) be the range of \( f \). By (1),
\[ R \subseteq [f(r), f(p)]. \]
By the IVT, there exists \( c \) in \( [r, p] \) or \( [p, r] \) such that \( f(c) = v \). Thus,
\[ v \in R \text{ and } [f(r), f(p)] \subseteq R. \]
Hence, \( R = [f(r), f(p)] \).

6) Let \( f : [a, b] \to [a, b] \) be continuous. Consider \( h(x) = f(x) - x \).
It suffices to show that there exists \( x_0 \in [a, b] \) such that
\[ h(x_0) = 0. \]
(Then, \( f(x_0) = x_0 \) and \( x_0 \) is a fixed point of \( f \).)
Observe, that since \( a \leq f(x) \leq b \) for all \( x \in [a, b] \), we have:
\[ h(a) = f(a) - a \geq 0 \quad \text{and} \quad h(b) = f(b) - b \leq 0. \]
If the equality holds in one of the above inequalities, we can
take \( x_0 = a \), or, respectively, \( x_0 = b \).
Suppose that both inequalities are sharp; that is:
\[ h(a) > 0 \quad \text{and} \quad h(b) < 0. \]
Since \( h \) is continuous in \([a, b]\), by the IVT there exists \( x_0 \in [a, b] \)
such that \( h(x_0) = 0 \).

1) We have to show that
\[
\forall \varepsilon > 0 \exists \delta > 0 \quad \text{s.t.} \quad |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.
\]
Let \( \varepsilon = \varepsilon_0 \). Let \( \delta > 0 \) be given. Take \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \delta \).
Let \( x_0 = n \), \( y_0 = (n + \frac{1}{n}) \). Then \( x_0, y_0 \in (0, +\infty) \),
\[
|\frac{n}{n} - \frac{n + 1}{n}| = \frac{1}{n} < \delta \quad \text{and} \quad |f(x_0) - f(y_0)| = (n + \frac{1}{n})^2 - n^2 = 2 + \frac{1}{n^2} > \varepsilon_0.
\]
Thus, \( f(x) = x^2 \) is not uniformly continuous on \((0, +\infty)\).
2) Theorem 20.1 does not apply directly as the interval \([1, +\infty)\) is not bounded. To prove that \(\frac{1}{x}\) is uniformly continuous on \([1, +\infty)\), we have to show that:

\[
\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in [1, +\infty) \quad |x - y| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon . \tag{2}
\]

Let \(\varepsilon > 0\) be fixed. Take \(K = \frac{1}{\sqrt{\varepsilon}}\). Since \(\frac{1}{x}\) is continuous on the interval \([1, K]\), \(\frac{1}{x}\) is uniformly continuous on \([1, K]\). Hence, there exists \(\delta_1 > 0\) such that

\[
\forall x, y \in [1, K] \quad (|x - y| < \delta_1 \Rightarrow \left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon) . \tag{3}
\]

Take \(\delta = \min\left\{ \frac{1}{2}, \delta_1 \right\}\). Then from (3)

\[
\forall x, y \in [1, K] \quad (|x - y| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon) . \tag{4}
\]

Since \(1 \geq \frac{1}{x} - \frac{1}{y} \geq \frac{1}{xy} \) and \(x > K\) and \(y > K\) implies \(\frac{1}{xy} < \varepsilon\), we have \(\frac{|x - y|}{xy} < \varepsilon\) whenever \(|x - y| < 1\) and \(x, y \in (K, +\infty)\). The latter and (4) give:

\[
\forall x, y \in [1, +\infty) \quad (|x - y| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{y} \right| < \varepsilon) .
\]

Thus \(g(x) = \frac{1}{x}\) is uniformly continuous on \([1, +\infty)\).

5) Is straightforward from Def 21.1 and Th 16.1.

3) Suppose \(f\) is not constant on \((-\infty, +\infty)\). Then for some \(a, b \in (-\infty, +\infty)\), we have \(f(a) \neq f(b)\). Without any loss of generality, we may assume \(a < b\). We have \(f(a) < f(b)\) or \(f(b) > f(a)\). From the density of irrationals there exists \(\nu\) between \(f(a)\) and \(f(b)\) such that \(\nu \in \mathbb{R} \setminus \mathbb{Q}\). As \(f\) is continuous on \([a, b]\), the Intermediate Value Theorem implies that for some \(c \in (a, b)\), \(f(c) = \nu\). Contradiction as \(f\) takes only rational values.