1) Assume first that $\{a_n\}_{n=1}^{\infty}$ is increasing. If $\{a_n\}_{n=1}^{\infty}$ is bounded, then for some $L \in \mathbb{R}$, $\lim a_n = L$ by Thm. 1 and the proposition is proved. Suppose that $\{a_n\}_{n=1}^{\infty}$ is not bounded. Since $\{a_n\}_{n=1}^{\infty}$ is increasing, it is bounded below by $a_1$. Hence, $\{a_n\}_{n=1}^{\infty}$ must be not bounded above. We shall show that

$$\lim_{n \to \infty} a_n = +\infty. \quad (1)$$

Indeed, take $N \in \mathbb{R}, M > 0$. Since $\{a_n\}_{n=1}^{\infty}$ is not bounded above, there exists $N \in \mathbb{N}$ such that $a_N > M$. As $\{a_n\}_{n=1}^{\infty}$ is increasing, $a_n \geq a_N > M$ for all $n > N$ and (1) is proved.

In the case when $\{a_n\}_{n=1}^{\infty}$ is decreasing, the proof is very similar so we skip it.

2) Assume $\{a_n\}_{n=1}^{\infty}$ is not bounded above. We shall define a subsequence $\{a_{m_n}\}_{n=1}^{\infty}$ such that $a_{m_n} \to +\infty$ as $n \to +\infty$ by induction. Since $\{a_n\}_{n=1}^{\infty}$ is not bounded above, there exists $m_1 \in \mathbb{N}$ such that $a_{m_1} > 1$.

Choose $m_2 \in \mathbb{N}$ such that

$$m_2 > m_1 \text{ and } a_{m_2} > 2.$$ 

Such $m_2$ exists. Indeed, suppose it doesn't. Then $a_n \leq 2$ for all $n > m_1$ and the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above by

$$\max \{1a_1, \ldots, 1a_{m_1}, 2\}$$

which contradicts our assumption. Suppose for a given $k \in \mathbb{N}$ we have defined $m_1, m_2, \ldots, m_k$ such that:

$$m_1 < m_2 < \ldots < m_k \quad \text{and} \quad a_{m_i} > i \quad \text{for} \quad i = 1, \ldots, k.$$ 

Choose $m_{k+1} > m_k$ and such $a_{m_{k+1}} > k + 1$. Such $m_{k+1} \in \mathbb{N}$ exists as otherwise $\{a_n\}_{n=1}^{\infty}$ would be bounded above by

$$\max \{1a_1, \ldots, 1a_{m_k}, k + 1\}.$$ 

By induction we can choose $m_{n}$ for all $n \in \mathbb{N}$ in such a way that:

$$m_1 < m_2 < \ldots < m_n < m_{n+1} < \ldots \quad \text{and} \quad a_{m_n} > n \quad \text{for all} \quad n \in \mathbb{N}.$$
Hence, \( \{a_n^2\}_{n=1}^{\infty} \) is a subsequence of \( \{a_n\}_{n=1}^{\infty} \). The condition \( a_n > n \) for all \( n \in \mathbb{N} \) implies easily that \( \lim_{n \to +\infty} a_n = +\infty \).

(Using Def 11.1.)

Prop 11.6.1: Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence which is not bounded below. Then \( \{a_n\}_{n=1}^{\infty} \) has a subsequence \( \{a_{n_k}\}_{k=1}^{\infty} \) such that \( \lim_{n \to +\infty} a_{n_k} = -\infty \).

3) Let \( \{x_n\}_{n=1}^{\infty} = 0, 1, 2, 3, \ldots \), \( \{y_n\}_{n=1}^{\infty} = 0, 0, 0, \ldots \). Then the set of elements \( \{y_n: n \in \mathbb{N}, y_n = 0, 0, 0, \ldots \} \) is contained in the set of elements \( \{x_n: n \in \mathbb{N}, x_n = 0, 0, 0, \ldots \} \). Yet, \( \{y_n\}_{n=1}^{\infty} \) is not a subsequence of \( \{x_n\}_{n=1}^{\infty} \).

Indeed, \( y_n = x_{m_n} \) for \( m_n = 1 \) for all \( n \). The sequence of indices \( \{m_n\}_{n=1}^{\infty} \) is not strictly increasing.

4) Since the problem does not say that the subsequences of \( \{\sin \frac{n\pi}{4}\}_{n=1}^{\infty} \) chosen must converge to different limits, it suffices to choose one convergent subsequence and then five subsequences of this subsequence. (A subsequence of a subsequence is a subsequence of the original sequence, a subsequence of a convergent sequence converges.) Take for example:

\[ \{\sin \frac{4n\pi}{4}\}_{n=1}^{\infty} = \{\sin (n\pi)\}_{n=1}^{\infty} = 0, 0, 0, \ldots \] and

\[ \{\sin \frac{8n\pi}{4}\}_{n=1}^{\infty}, \{\sin \frac{12n\pi}{4}\}_{n=1}^{\infty}, \{\sin \frac{16n\pi}{4}\}_{n=1}^{\infty}, \{\sin \frac{20n\pi}{4}\}_{n=1}^{\infty}, \{\sin \frac{24n\pi}{4}\}_{n=1}^{\infty}. \]

5) Let \( \{a_n\}_{n=1}^{\infty} \) be an unbounded sequence. By Problem 1, there exists a subsequence \( \{a_{m_n}\}_{n=1}^{\infty} \) such that \( \lim_{n \to +\infty} a_{m_n} = +\infty \) or there exists a subsequence \( \{a_{p_n}\}_{n=1}^{\infty} \) such that \( \lim_{n \to +\infty} a_{p_n} = -\infty \).
Easily we prove that if such \( \{a_n \}_{n=1}^{\infty} \) exists, then
\[
\lim_{n \to \infty} a_n = 0.
\]
(See H5, Pr. 2.). Thus if such \( \{a_n \}_{n=1}^{\infty} \) exists, then
\[
\lim_{n \to \infty} a_n = 0.
\]

(6) Suppose that the sequence \( \{x_n\}_{n=1}^{\infty} \) does not converge to \( L \). Then
\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \quad |x_n - L| \geq \varepsilon.
\]
(2)

Take and fix \( \varepsilon_0 \) which satisfies (2). Applying (2) with \( N = 1 \), choose \( m_1 \in \mathbb{N} \) such that \( |x_{m_1} - L| \geq \varepsilon_0 \). Applying (2) with \( N = m_1 + 1 \) we can choose \( m_2 \in \mathbb{N} \) such that
\[
m_2 > m_1 \quad \text{and} \quad |x_{m_2} - L| \geq \varepsilon_0.
\]
Applying (2) with \( N = m_2 + 1 \), choose \( m_3 \in \mathbb{N} \) such that
\[
m_3 > m_2 > m_1 \quad \text{and} \quad |x_{m_3} - L| \geq \varepsilon_0.
\]
And so on, there exists a sequence \( m_n \), \( n = 1, 2, \ldots \) such that:
\[
m_1 < m_2 < \ldots < m_n < m_{n+1} < \ldots \quad \text{and} \quad |x_{m_n} - L| \geq \varepsilon_0.
\]
(3)
(Of course, more precisely we are defining \( \{x_{m_n}\}_{n=1}^{\infty} \) by induction as in Problem 2.) By (3), \( \{x_{m_n}\}_{n=1}^{\infty} \) is a subsequence of \( \{x_n\}_{n=1}^{\infty} \).

Since \( \{x_n\}_{n=1}^{\infty} \) was assumed bounded, \( \{x_{m_n}\}_{n=1}^{\infty} \) is bounded.

By Bolzano-Weierstrass theorem \( \{x_{m_n}\}_{n=1}^{\infty} \) has a convergent subsequence \( \{x_{m_{p_n}}\}_{n=1}^{\infty} \). A subsequence of a subsequence of \( \{x_n\}_{n=1}^{\infty} \) is a subsequence of \( \{x_n\}_{n=1}^{\infty} \). Hence, \( \{x_{m_{p_n}}\}_{n=1}^{\infty} \) is a convergent subsequence of \( \{x_n\}_{n=1}^{\infty} \) and by (3)
\[
|x_{m_{p_n}} - L| \geq \varepsilon_0 \quad \text{for all} \quad n \in \mathbb{N}.
\]
Thus, \( \{x_{m_{p_n}}\}_{n=1}^{\infty} \) does not converge to \( L \). Contradiction as every convergent subsequence of \( \{x_n\}_{n=1}^{\infty} \) must converge to \( L \). Therefore, \( \{x_n\}_{n=1}^{\infty} \) converges to \( L \).