Edge-Rooted Forests and $\alpha$-Invariant of Cone Graphs

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Abstract. We define the $\alpha$-invariant of a finite graph $G$ to be $\alpha(G) := T_{M(G)}(0, 1)$ where $T_{M(G)}(x, y)$ is the Tutte polynomial of the cycle matroid $M(G)$ of $G$. The cone $\hat{G}$ on a graph $G$ is obtained by adjoining a new vertex $p$ and then joining each vertex of $G$ to $p$ by a single edge. In this paper we show that $\alpha(\hat{G})$ is the cardinality of the set of all edge-rooted forests in the base graph $G$. We will apply this result to compute the $\alpha$-invariants of the complete graphs $K_n$ and the wheels $W_n$ where a wheel is a cone on a circuit.

Key words. Tutte-Grothendieck invariants, cycle matroid of graphs, edge-rooted forests

1. Introduction

Given a finite graph $G$, let $T_{M(G)}(x, y)$ be the Tutte polynomial of its cycle matroid $M(G)$. We define the $\alpha$-invariant of $G$ to be Tutte-Grothendieck invariant of $M(G)$ given by the evaluation of $T_{M(G)}(x, y)$ at $(x, y) = (0, 1)$, i.e., $\alpha(G) := T_{M(G)}(0, 1)$. Recall that for a matroid $M$ in general, $T_M(0, 1)$ equals the unsigned reduced Euler number of the independence complex of $M$. (Refer to [2] for Tutte polynomials in general and [1] for definitions and results concerning matroid complexes.) Hence if $G$ is connected and has $n + 1$ vertices, and if we let $f_i$ be the number of spanning forests with $i$ edges in $G$, then we may also define $\alpha(G)$ as

$$\alpha(G) := \sum_{0 \leq i \leq n} (-1)^{n-i} f_i.$$

In this paper we will examine $\alpha$-invariants of cone graphs (see section 3 for definition). For example, the complete graph $K_{n+1}$ ($n \geq 1$) is the cone on $K_n$ and the wheel $W_n$ of order $n$ ($n \geq 1$) is the cone on the circuit $C_n$ of length $n$. Previously, $\alpha(K_n)$ has been considered by Kook [3] in 1996 and by Novik, Postnikov, and Sturmfels [4] in 2001. In [3], $\alpha(K_n)$ was computed as the rank of the reduced homology group for the independence complex of $M(K_n)$. In [4], $\alpha(K_n)$ was described in terms of the Hermite polynomials, a generating function for partial matchings (degree 1 subgraphs) in a graph.

The main theorem of this paper (Theorem 1. in section 3) is a simple and elegant combinatorial interpretation for the $\alpha$-invariant of cones graphs in general; for the cone $\hat{G}$ on any finite graph $G$, $\alpha(\hat{G})$ is the number of edge-rooted forests in $G$ (see section 2 for definition). Moreover, the proof of the main theorem is purely graph theoretic and combinatorial, and we will see that this interpretation arises naturally from the above
definition of α-invariants of graphs as an alternating sum. As a corollary we obtain a combinatorial interpretation for α(K_{n+1}) as the number of edge-rooted forests in K_n.

In the last section we will apply these results to derive a formula for α(W_n) and the exponential generating function for α(K_n). In this paper we consider finite graphs only.

2. Edge-rooted forests

Recall that a forest \( F \) in a graph \( G = (V(G), E(G)) \) is a subgraph of \( G \) such that each component of \( F \) is a tree. A forest \( F \) in \( G \) is called spanning if \( V(F) = V(G) \). We will denote the set of all spanning forests in \( G \) by \( \mathcal{I}(G) \), which may also be regarded as the set of independent sets of the cycle matroid of \( G \). (Refer to \([5]\) for the definition of cycle matroid of graphs.) The rank of \( F \in \mathcal{I}(G) \) is the number \( |E(F)| \) of edges in \( F \).

Given \( F \in \mathcal{I}(G) \), let \( C(F) \) denote the set of its components. In particular every \( T \in C(F) \) is a tree. Note that \( T \) may be an isolated vertex. We define a spanning forest \( F \) in \( G \) to be none-isolated if each component \( T \in C(F) \) has at least one edge, that is, no \( T \in C(F) \) is an isolated vertex. We will denote the set of all none-isolated spanning forests in \( G \) by \( \hat{\mathcal{I}}(G) \).

**Definition.** For each \( F \in \hat{\mathcal{I}}(G) \) let \( E^*(F) := \prod_{T \in C(F)} E(T) \). The set of edge-rooted forests in \( G \), written \( \hat{\mathcal{F}}(G) \), is the following set of pairs:

\[
\hat{\mathcal{F}}(G) := \{(F, e^*) | F \in \hat{\mathcal{I}}(G) \text{ and } e^* \in E^*(F)\}.
\]

Given an edge-rooted forest \((F, e^*)\), \( F \) is called the support and \( e^* \) the edge-roots.

In other words, an edge-rooted forest in \( G \) is a none-isolated spanning forest \( F \) in \( G \) with one edge from each \( T \in C(F) \) marked as an edge-root.

It is clear that the number of edge-rooted forests with a given support \( F \in \hat{\mathcal{I}}(G) \) is \( |E^*(F)| \). Now suppose \( |C(F)| = m \). Since \( |E(T)| = |V(T)| - 1 \) when \( T \) is a tree, we have

\[
|E^*(F)| = \prod_{T \in C(F)} |E(T)| = \prod_{T \in C(F)} (|V(T)| - 1) = \sum_{n=0}^{m} (-1)^{m-n} \nu_n(F),
\]

where \( \nu_n(F) \) denotes the evaluation of \( n \)-th elementary symmetric polynomial in \( m \) variables with the \( m \) values \( |V(T)| \) for \( T \in C(F) \).

**Definition.** Let \( F \in \mathcal{I}(G) \) and suppose \( C(F) = m \). The \( n \)-th vertex configuration space \( V^*(F)^{(n)} \) of \( F \) (\( 0 \leq n \leq m \)) is the set of all \( n \)-subsets \( \{v_1, \ldots, v_n\} \subset V(F) \) such that \( v_i \) and \( v_j \) do not belong to the same component of \( F \) if \( i \neq j \). Equivalently, we define

\[
V^*(F)^{(n)} := \bigcup_{A \subset C(F), |A| = n} \left( \prod_{T \in A} V(T) \right),
\]

where the disjoint union is taken over all subsets of \( C(F) \) of cardinality \( n \).

Note that \( V^*(F)^{(0)} \) is a set with a single element, the empty set, hence \( |V^*(F)^{(0)}| = 1 \). It is also clear that \( |V^*(F)^{(n)}| = \nu_n(F) \) for all \( 0 \leq n \leq m \). Now we summarize the above discussion in the following lemma whose proof is clear.
Lemma 1. Let $F \in \mathcal{I}(G)$ and suppose $|C(F)| = m$. Then the number of the edge-rooted forests supported by $F$ is given by
\[
|E^*(F)| = \prod_{T \in C(F)} (|V(T)| - 1) = \sum_{0 \leq n \leq m} (-1)^{m-n}|V^*(F)^{(n)}|.
\]
Hence $|E^*(F)|$ is non-zero if and only if $F$ is none-isolated. \hfill \Box

3. \(\alpha\)-invariant of cone graphs

Definition. The cone graph, or simply the cone, on a finite graph $G$ is a graph $\hat{G}$ obtained from $G$ by adjoining a new vertex $p$ called the cone point and then joining each vertex $v \in V(G)$ to $p$ by a single edge. We will call $G$ the base of $\hat{G}$. For example, the complete graph $K_{n+1}$, the wheel $W_n$ of order $n$, and the fan $\hat{P}_n$ of order $n$ are cones on the bases $K_n$, the circuit $C_n$ of length $n$, and the path $P_n$ of length $n$, respectively ($n \geq 1$).

For a cone $\hat{G}$, the base $G$ is naturally a subgraph of $\hat{G}$. Also the star $S(p)$ of $p$ in $\hat{G}$, i.e., the cone on $V(G)$ with the cone point $p$, is a subgraph of $\hat{G}$. Clearly $E(G) \cap E(S(p)) = \phi$, and the following observation about $\hat{G}$ will be important:
\[
\hat{G} = G \cup S(p).
\]
Accordingly, for any $F' \in \mathcal{I}(\hat{G})$, we have a decomposition $F' = (F' \cap G) \cup (F' \cap S(p))$; we will call $F' \cap G$ the support of $F'$ and $F' \cap S(p)$ the suspenders of $F'$, denoted by $S(p,F')$. The support of $F'$ is a spanning forest in $G$ and $S(p,F')$ is the star of $p$ in $F'$.

Clearly the cone $\hat{G}$ on any $G$ is connected and the rank of its cycle matroid $M(\hat{G})$ is $r = |V(G)| = |V(F)|$ for any $F \in \mathcal{I}(G)$. Given an integer $i \in [0,r]$ and $F \in \mathcal{I}(G)$, let $\mathcal{I}_i$ be the set of all rank $i$ spanning forests in $\hat{G}$, and let $\mathcal{I}_{F,i}$ be the subset of $\mathcal{I}_i$ consisting of those with the support $F$. Then we have $f_i = |\mathcal{I}_i|$ and define $f_{F,i} := |\mathcal{I}_{F,i}|$. Since the set $\{\mathcal{I}_{F,i} | F \in \mathcal{I}(G)\}$ partitions $\mathcal{I}_i$ for any $i \in [0,r]$, we have
\[
f_i = \sum_{F \in \mathcal{I}(G)} f_{F,i}.
\]
Note that if $i < |E(F)|$, then $\mathcal{I}_{F,i}$ is empty and $f_{F,i} = 0$. The following lemma provides a crucial link between edge-rooted forests in $G$ and $\alpha(\hat{G})$.

Lemma 2. Fix $F \in \mathcal{I}(G)$ and $i \in [|E(F)|,|V(F)|]$. For $n = i - |E(F)|$, we have
\[
f_{F,i} = |V^*(F)^{(n)}|.
\]
Proof. We will construct a bijection $\phi : \mathcal{I}_{F,i} \rightarrow V^*(F)^{(n)}$ for $n = i - |E(F)|$. Note that $F' \in \mathcal{I}_{F,i}$ is determined by its suspender $S(p,F')$ because its support $F'$ is given. The number of edges in $S(p,F')$ is $|E(F') \setminus E(F)| = i - |E(F)| = n$, which is also the degree of $p$ in $F'$. Furthermore no two edges in $S(p,F')$ are connected to the same component of the base $F'$ because $F'$ is a forest. Therefore the set of vertices in $F'$ that are adjacent to $p$, which we denote by $N(p,F')$, is a subset of $V(F)$ of cardinality $n$ such that no two vertices in $N(p,F')$ belong to the same component of $F$. Therefore $N(p,F') \in V^*(F)^{(n)}$. Conversely, given any $N \in V^*(F)^{(n)}$, it’s easy to see that $F \cup \hat{N}$, where $\hat{N}$ is a cone on $N$ with $p$ as the cone point, is a spanning forest in $\hat{G}$ with $i$ edges and $F$ as the base. Therefore the mapping $\phi : \mathcal{I}_{F,i} \rightarrow V^*(F)^{(n)}$ given by $\phi(F') = \hat{N}(p,F')$ is a bijection with its inverse $\psi : V^*(F)^{(n)} \rightarrow \mathcal{I}_{F,i}$ given by $\psi(N) = F \cup \hat{N}$. The proof is complete. \hfill \Box
We make important remarks on the range of \( i \) and \( n = i - |E(F)| \) in the above lemma. When \( i = |E(F)| \), we have \( n = 0 \) and \( I_{F,i} \) consists of \( F \) only. Therefore \( f_{F,i} = |V^*(F)^{(n)}| = 1 \). Now recall that the number of components of any forest \( F \) in general is given by \( |V(F)| - |E(F)| \). Therefore, when \( i = |V(F)| \), we have \( n = |C(F)| \), and \( V^*(F)^{(n)} \) is well-defined. Moreover, Since the rank of \( \mathcal{M}(\hat{G}) \) is \( |V(F)| \), \( f_{F,i} = 0 \) for all \( i \geq |V(F)| \).

Now we are ready to prove the main theorem of the paper.

**Theorem 1.** Let \( \hat{G} \) be the cone on a finite graph \( G \). Then \( \alpha(\hat{G}) \) is the number of edge-rooted forests in the base \( G \):

\[
\alpha(\hat{G}) = |\dot{\mathcal{F}}(G)|.
\]

**Proof.** Let \( r = |V(G)| = \text{rank of } \mathcal{M}(\hat{G}) \). Then we have

\[
\alpha(\hat{G}) = \sum_{0 \leq i \leq r} (-1)^{r-i} f_i
= \sum_{0 \leq i \leq r} (-1)^{r-i} \sum_{F \in \mathcal{I}(G)} f_{F,i}
= \sum_{0 \leq i \leq r} \sum_{F \in \mathcal{I}(G)} (-1)^{r-i} f_{F,i}
= \sum_{F \in \mathcal{I}(G)} \sum_{|E(F)| \leq i \leq r} (-1)^{r-i} f_{F,i},
\]

where the last equality is from the fact \( f_{F,i} = 0 \) when \( i < |E(F)| \) for all \( F \in \mathcal{I}(G) \). Now if we let \( n = i - |E(F)| \), then by Lemma 2, we have \( f_{F,i} = |V^*(F)^{(n)}| \), and we have \( r - i = |V(G)| - n - |E(F)| = |C(F)| - n \) by the remarks following Lemma 2. Therefore if we let \( m_F = |C(F)| \) for every \( F \in \mathcal{I}(G) \), then by Lemma 1. we have,

\[
\alpha(\hat{G}) = \sum_{F \in \mathcal{I}(G)} \sum_{0 \leq n \leq m_F} (-1)^{m_F-n}|V^*(F)^{(n)}| = \sum_{F \in \mathcal{I}(G)} |E^*(F)| = |\dot{\mathcal{F}}(G)|.
\]

The proof of the theorem is complete. \( \square \)

4. Examples: \( \alpha \)-invariants of wheels and complete graphs

Recall that a wheel \( W_n \) of order \( n \) is a cone on a circuit \( C_n \) of length \( n \).

**Theorem 2.** Let \( W_n \) be a wheel of order \( n \) (\( n \geq 1 \)). Then \( \alpha(W_n) = 2^n - 2 \).

**Proof.** Since \( W_n \) is a cone on a circuit \( C_n \) of length \( n \), it suffices to show that the number of edge-rooted spanning forests in \( C_n \) is \( 2^n - 2 \). Note that edge-rooted spanning forests in \( C_n \) can be constructed in two steps as follows. First pick a positive even number of edges from \( C_n \) and assign to each of these edges a plus or minus sign in such a way that the signs will alternate as one goes around the circuit \( C_n \). Then we will get two edge-rooted spanning forests in \( C_n \), first one by marking those edges with plus signs as edge-roots and deleting those with minus signs, and then the second by doing the same thing with the signs switched. It’s clear that every edge-rooted forest in \( C_n \) can be obtained this way. Now since an \( n \)-set has \( 2^{n-1} - 1 \) non-empty subsets of even cardinality and each of these subsets gives rise to two edge-rooted spanning forests in \( C_n \), the result follows. \( \square \)

**Remark.** If we pick an odd number of edges from \( C_n \) and try to construct an edge-rooted spanning forest as in the above proof, the outcome will be a forest with one of
the components having either no edge-root or two edge-roots. However, for a path \( P_n \) of length \( n \), a set of odd number of edges in \( P_n \) will correspond to an edge-rooted forest in \( P_n \); assign alternating signs to these edges starting with a plus and then mark those with plus signs as roots and delete those with minus signs as in the above proof. Therefore for the fan \( \tilde{P}_n \) of order \( n \), we have \( \alpha(\tilde{P}_n) = 2^{n-1} \) the number of odd-sized subsets of an \( n \)-set.

**Theorem 3.** Let \( K_n \) be the complete graph on \( n \) vertices \((n \geq 1)\). Then \( \alpha(K_1) = 1 \) and \( \alpha(K_{n+1}) \) is the number of edge-rooted forests in \( K_n \) for \( n \geq 1 \).

**Proof.** \( \alpha(K_1) = 1 \) is clear because for \( K_1 \) we have \( f_0 = 1 \) and \( f_i = 0 \) for all \( i > 0 \). Since \( K_{n+1} \) is a cone on \( K_n \) \((n \geq 1)\), the second statement of the theorem follows immediately from Theorem 1. \( \square \)

The exponential generating function for \( \alpha(K_{n+1}) \) \((n \geq 0)\) can be derived from this theorem as follows. A typical edge-rooted forest in \( K_n \) is obtained by choosing a partition \( \{B_1, B_2, \ldots, B_t\} \) of the vertex set \([n]\) \(\text{(in particular } |B_i| \geq 1 \text{ for all } i)\), and then constructing an edge-rooted tree in each block \( B_i \) \((1 \leq i \leq t)\). Therefore if we let \( \bar{\tau}(m) = (m-1)m^{m-2} \), which counts the number of edge-rooted trees on \( m \) vertices, we have

\[
\alpha(K_{n+1}) = \sum_{\{B_1, B_2, \ldots, B_t\} \vdash [n]} \bar{\tau}(|B_1|)\bar{\tau}(|B_2|)\cdots \bar{\tau}(|B_t|),
\]

where the sum ranges over all partitions \( \{B_1, B_2, \ldots, B_t\} \) of the set \([n]\). Since \( \alpha(K_1) = 1 \), it follows from [6, Theorem 5.1.6] that the exponential generating function for \( \alpha(K_{n+1}) \) \((n \geq 0)\) is

\[
\sum_{n \geq 0} \frac{\alpha(K_{n+1})}{n!} x^n = \exp(T(x)),
\]

where \( T(x) = \sum_{m \geq 2} \bar{\tau}(m)x^m/m! \).

Finally we want to make an interesting remark about the magnitude of \( \alpha(K_n) \): it has been observed, but not proved, that the ratio \( \alpha(K_n)/(n^{n-2}) \) approaches \( e^{-1/2} \) monotonically from below as \( n \) grows.

**References**


