MTH 142 Practice Problems for Exam 2 - Fall 2001

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Sections 8.1, 8.2, 8.3, 9.1, 9.2, 9.3, 9.4, 9.5 and series handout (Skip p. 393)

1. a) Obtain the first three nonzero terms of the Taylor series of \( f(x) = e^{-x^2} \) about \( a = 0 \).
   b) Use part (a) to determine which of the functions \( f(x) = e^{-x^2} \) and \( g(x) = 1 - 3x^2 \) is larger for values of \( x \) that are close to zero.

2. a) Obtain \( P_3(x) = \) the Taylor polynomial of order 3 of \( \tan x \) about \( a = \pi/4 \)
   b) Use the plots of \( f(x) \) and \( P_3(x) \) to obtain an approximate value of the maximum error \( |f(x) - P_3(x)| \) for \( 0.7 \leq x \leq 1.2 \).

3. a) Calculate the radius of convergence of the series \( \sum_{n=0}^{\infty} n3^n x^n \).
   b) Sketch in the number line an open interval of points \( x \) for which the series converges. (No analysis of the endpoints of the interval is requested).

4. a) Calculate the radius of convergence of the series \( \sum_{n=0}^{\infty} \frac{n + 1}{n + 3} x^n \).
   b) Sketch in the number line an open interval of points \( x \) for which the series converges. (No analysis of the endpoints of the interval is requested).

5. A solid \( S \) is produced by revolving about the \( x \)-axis the region \( R \) in the plane bounded by \( y = 1, \ y = 2x^2, \) and \( x = 0 \).
   a) Write down a Riemann sum that approximates the volume of the solid.
   b) Find the exact volume of the resulting solid.

6. Answer questions (a) and (b) of the previous problem, only now the solid is produced by revolving \( R \) about the \( y \)-axis.

7. Answer questions (a) and (b) of the previous problem, only now the solid is produced by revolving \( R \) about the line \( x = -1 \).

8. An oil slick has the shape of a circle with radius 7,000 feet. After measurements were taken, it has been determined that the density of the oil at "r" feet from the center of the circle is given by the formula

\[ d(r) = \frac{0.004}{1 + r^2} \text{ Kg/ft}^2 \]

a) Write down a Riemann sum that approximates the total mass of the oil.
   b) Use part (a) to express the mass of the oil as an integral.

9. A 1 m. long rod has (linear) density \( d(x) = 2.0 + 0.015x \text{ gr/m,} \) where \( x \) is the distance from one end.
   a) Obtain a Riemann sum that approximates the total mass of the rod.
   b) Calculate the total mass of the rod as an integral.
   c) Calculate the center of mass of the rod.
10. A cylindrical container has as base a circle with radius 2 ft. and has height 3 ft. The container is filled with water containing a kind of bacteria that seems to accumulate near the bottom of the container, where more food is found. A researcher took samples and determined that the density of bacteria is a function of the distance \( y \) to the bottom in inches, as follows:

\[
d = 3.5 - 0.05y \quad \text{million bacteria per cubic inch}
\]

Determine the total number of bacteria in the container.

11. A 20 feet tall water tank has the shape of an inverted cone (i.e., the vertex is at the bottom) with circular top with radius 10 feet and height 20 feet. Water weighs 62.4 lbs/ft\(^3\).

a) Write down a Riemann sum that approximates the work required to take the water out of the tank from the top.

b) Obtain the exact work in part (a) by calculating a suitable integral.

12. A dam has the shape of a trapezoid, with horizontal parallel sides measuring 30 ft. (bottom) and 60 ft (top). The height of the dam is 30 ft., and one vertical side is perpendicular to both base and top. The dam has water up to the top on one side. (Water weighs 62.4 lbs/ft\(^3\).)

a) Write down a Riemann sum that approximates the total force exerted by the water on the dam.

b) Obtain an integral that gives the total force of the water on the dam.

13. A certain amount of fresh water shrimp is placed in a tank together with 2 lbs. of food, at 12:00 p.m. on January 1st. An additional 2 lbs. of food are added to the tank every 24 hours. After every 24 hours, 85% of the food either decomposes or is eaten. How much food is in the tank right before 12:00 p.m. on January 20th? Give details, and explain how you arrived at your answer.

14. a) Plot the \( 2\pi \) periodic function \( f(x) \) given by \( f(x) = |x| \), for \( -\pi < x \leq \pi \). For the plot, use \(-3\pi \leq x \leq 3\pi\), and \( 0 \leq y \leq 2\pi \).

b) Calculate the Fourier polynomial of order 2 of \( f(x) \). It is ok to compute the integrals that appear in the coefficients with a calculator (you may use Simpson’s rule with \( n = 20 \), for example. Absolute value may be entered in your calculator as sqrt( \( x^2 \))

**Determine if the series given below converges or diverges**

(Note: each one can be treated in more than one way!)

15. \[ \sum_{n=0}^{\infty} \frac{n^2}{1 + n^3} \]

16. \[ \sum_{n=1}^{\infty} \frac{1}{2^n} \]

17. \[ \sum_{n=3}^{\infty} \frac{(-2)^{n+1}}{\pi^n} \]
SOLUTION MTH142 Practice Problems for Exam 2

1. a) Take \( y = -x^2 \) in \( e^y = 1 + \frac{1}{1!}y + \frac{1}{2!}y^2 + \cdots \) to get

\[
e^{-x^2} = 1 + \frac{1}{1}(-x^2) + \frac{1}{2}(-x^2)^2 + \cdots = 1 - x^2 + \frac{1}{2}x^4 + \cdots
\]

This gives the answer in less time than actually calculating the coefficients one by one (which is ok too).

b) For small \( x \) we may drop terms of higher order in the Taylor series from part (a) to have \( e^{-x^2} \approx 1 - \frac{1}{2}x^2 \), which is clearly larger than \( g(x) = 1 - \frac{3}{2}x^2 \). So \( f(x) \) is the larger function of the two, for small values of \( x \).

2. a) Direct calculation gives the polynomial

\[
1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3
\]

b) From the plot we see that the gap between \( f(x) \) and \( p(x) \) is the largest possible at the endpoint \( x = 1.2 \). Therefore the maximum error in the interval \((0.7, 1.2)\) is \( D = f(1.2) - p(1.2) = 0.20911 \).

3. a) The radius of convergence is given by

\[
r = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \to \infty} \frac{n3^n}{(n + 1)3^{n+1}} = \lim_{n \to \infty} \frac{n}{n + 1} = \frac{1}{3}
\]

b) The interval is \(-\frac{1}{3} < x < \frac{1}{3}\)

4. a) The radius of convergence is given by

\[
r = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \to \infty} \frac{n+1}{n+2} = \lim_{n \to \infty} \frac{(n+1)(n+4)}{(n+2)(n+3)} = \lim_{n \to \infty} \frac{n^2 + 5n + 4}{n^2 + 5n + 6} = \lim_{n \to \infty} \frac{1 + \frac{5}{n} + \frac{4}{n^2}}{1 + \frac{5}{n} + \frac{6}{n^2}} = 1
\]

b) The interval is \(-1 < x < 1\)

5. a) By taking sections perpendicular to the axis of rotation, we get “washers”. At the tickmark \( x_j \) the washer has inner radius \( r_j = 2x_j^2 \), outer radius \( R_j = 1 \), and thickness \( \Delta x \).

The Riemann sum that approximates the volume is

\[
V \approx \sum_{j=0}^{n} (\pi R_j^2 - \pi r_j^2) \Delta x = \sum_{j=0}^{n} (\pi 1^2 - \pi (2x_j^2)^2) \Delta x
\]

The volume is obtained by taking limit as \( \Delta x \to 0 \). We have,

\[
Vol(S) = \int_{0}^{\sqrt{2}/2} (\pi - \pi 4x^4)dx = \frac{2\sqrt{2}\pi}{5} \approx 1.777153
\]
a) By taking sections perpendicular to the axis of rotation, we get “disks”. At the tickmark \( y_j \) the radius \( r_j = x_j = \sqrt{y_j/2} \) and thickness \( \Delta y \).

The Riemann sum that approximates the volume is

\[
\sum_{j=0}^{n} \pi R_j^2 \Delta y = \sum_{j=0}^{n} \pi (\sqrt{y_j/2})^2 \Delta y = \sum_{j=0}^{n} \pi y_j/2 \Delta y
\]

b) The volume is obtained by taking limit as \( \Delta y \to 0 \). We have,

\[
Vol(S) = \int_{0}^{1} \frac{\pi}{2} dy = \frac{\pi}{4}
\]

7. a) By taking sections perpendicular to the axis of rotation, we get “washers”. At the tickmark \( y_j \) the washer has inner radius \( r_j = 1 \), outer radius \( R_j = 1 + \sqrt{y_j/2} \), and thickness \( \Delta y \). The Riemann sum that approximates the volume is

\[
V \approx \sum_{j=0}^{n} (\pi R_j^2 - \pi r_j^2) \Delta y = \sum_{j=0}^{n} (\pi (1 + \sqrt{y_j/2})^2 - \pi (1)^2) \Delta y
\]

The volume is obtained by taking limit as \( \Delta x \to 0 \). We have,

\[
Vol(S) = \int_{0}^{1} (\pi (1 + \sqrt{y/2})^2 - \pi) dy = \pi (\frac{1}{4} + \frac{2\sqrt{2}}{3}) \approx 3.74732
\]

8. a) \[
\sum_{j=0}^{n} \frac{0.004}{1 + r_j^2} 2\pi r \Delta r
\]

b) \[
\int_{0}^{7000} \frac{(0.004) 2\pi r}{1 + r^2} \Delta r
\]

9. a) \[
\sum_{j=0}^{n} (2 + 0.015x) \Delta x
\]

b) \[
\int_{0}^{1} (2 + 0.015x) dx = 2.0075
\]

c) \[
\pi = \frac{\int_{0}^{1} x (2 + 0.015x) dx}{\int_{0}^{1} (2 + 0.015x) dx} = \frac{1.0050}{2.0075} = 0.50062266
\]
10. Slice the drum with horizontal, circular sections. Each section corresponds to a tickmark $y_j$ on the vertical axis. The number of bacteria in a section at tickmark $y_j$ is

$$\text{Number}(\text{Section}_j) \approx \text{density} \cdot \text{volume} = (3.5 - 0.05y_j) \pi (24)^2 \Delta y$$

The total number of bacteria is approximated by the Riemann Sum

$$\text{Number(Container)} = \sum_{j=1}^{n} (3.5 - 0.05y_j) \pi (24)^2 \Delta y$$

The exact number is obtained by passing to the limit as $\Delta y \to 0$. It is,

$$\int_0^{36} (3.5 - 0.05y) \pi (24)^2 dy = 169375 \text{ million bacteria}$$

A cross-section of the cone (shown in the figure) is bounded by the lines $y = \pm 2\pi$ and $y = 20$. Introduce tick marks in the $y$-axis.

11. The slab $S_j$ at height $y_j$ is a disk with radius $R_j = x_j = y_j/2$ and thickness $\Delta y$, so its volume is $\pi (y_j/2)^2 \Delta y$, and its weight is $62.4 \pi (y_j/2)^2 \Delta y$. The work involved in raising the slab a distance of $(20 - y_j)$ to the top of the cone is

$$w_j = (20 - y_j) 62.4 \pi (y_j/2)^2 \Delta y$$

The total work is approximated by

$$W \approx \sum_{j=0}^{n} (20 - y_j) 62.4 \pi (y_j/2)^2 \Delta y$$

The exact work is given by

$$\int_0^{20} (20 - y) 62.4 \pi (y/2)^2 \Delta y = 653451.2720$$

a) A sketch of the dam is shown in the figure above. Note that the equation of the right hand, non-horizontal side is $y = x - 30$. Introduce tick marks $y_0, y_1, \ldots, y_n$, in the $y$ axis.
At height $y_j$, the slab has area $(y_j+30)\Delta y$, and the pressure at this height is $62.4(30-y_j)$. Therefore the force on the slab is

$$F_j = 62.4(30-y_j)(y_j + 30)\Delta y$$

The total force is approximated by

$$F \approx \sum_{j=0}^{n} 62.4(y_j + 30)(30-y_j)\Delta y$$

The exact value of the total force is obtained by taking the limit as $\Delta y \to 0$:

$$F = \int_{0}^{30} 62.4(y_j + 30)(30-y_j)dy = 1,123,200$$

13. The following table is helpful:

<table>
<thead>
<tr>
<th>day</th>
<th>amount before 24hrs are up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan 2</td>
<td>(0.15) 2</td>
</tr>
<tr>
<td>Jan 3</td>
<td>(0.15) $2 + (0.15)^2$ 2</td>
</tr>
<tr>
<td>Jan 4</td>
<td>(0.15) $2 + (0.15)^2$ 2 + $(0.15)^3$ 2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Jan 20</td>
<td>$(0.15)^2$ 2 + $(0.15)^2$ 2 + ... + $(0.15)^{19}$ 2</td>
</tr>
</tbody>
</table>

Then, the amount right before noon on January 20th is

$$(0.15) 2 + (0.15)^2 2 + \cdots + (0.15)^{19} 2 = \frac{(0.15)2(1-(0.15)^{19})}{1-0.15} \approx 0.35294117647$$

14. a)  

b) we have that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{\pi}{4}$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(x) dx = \frac{1}{4}$$

$$a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(2x) dx = 0$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(x) dx = 0$$

$$b_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(2x) dx = 0$$
For details on hand-calculation, we calculate $a_1$ step by step below.

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(x) \, dx$$

$$= \frac{1}{\pi} \left( \int_{-\pi}^{0} -x \cos(x) \, dx + \int_{0}^{\pi} x \cos(x) \, dx \right)$$

$$= \frac{1}{\pi} \left( ( -x \sin(x) - \cos(x) ) \big|_{0}^{\pi} (x \sin(x) + \cos(x)) \big|_{0}^{\pi} \right)$$

$$= \frac{1}{\pi} \left( 2 + 2 \right)$$

Therefore, the Fourier polynomial we seek is $F_2(x) = \frac{x}{2} - \frac{1}{\pi} \cos(x)$.

15. Solution 1: Note that $\frac{n^2}{1+n^3}$ behaves like $\frac{1}{n}$ when $n$ is large, so we suspect that the series diverges. The following inequalities are clearly valid:

$$0 \leq \frac{n^2}{n^3 + n^3} \leq \frac{n^2}{1 + n^3}, \quad n = 1, 2, \ldots$$

The term in the center simplifies to $\frac{1}{2n}$. Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{n^2}{1+n^3}$. This is our original series with only the first term changed, so our original series $\sum_{n=0}^{\infty} \frac{n^2}{1+n}$ also diverges.

Solution 2: We use the integral test. Set $f(x) = \frac{x^2}{1+x^3}$ for $x > 1$. Now $\int_{1}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x^2}{1+x^3} \, dx = \lim_{b \to \infty} \ln |1 + x^3|_1^b = \lim_{b \to \infty} \frac{1}{3} \ln |1 + b^3| - \frac{1}{3} \ln 2 = \infty$ Therefore, the series diverges too.

16. Solution 1: Note that $\lim_{n \to \infty} (1 + \frac{1}{2n}) = 1 \neq 0$. Therefore the series diverges.

Solution 2: Using comparison, we have

$$0 \leq 1 \leq 1 + \frac{1}{2n}, \quad n = 0, 1, \ldots$$

Since $\sum_{n=0}^{\infty} 1$ is divergent, so is our original series.

17. Solution 1:

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{\pi^n} = \sum_{n=0}^{\infty} -2 \left( \frac{-2}{\pi} \right)^n = \text{a convergent geometric series, since } |x| = \left| -2/\pi \right| < 1$$

Solution 2: Since $\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{\pi^n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{\pi^n}$ the series is of the alternating (sign) type. To apply Leibnitz test we must first check two things. The first is,

$$\frac{2^1}{\pi^0} \geq \frac{2^2}{\pi^1} \geq \frac{2^3}{\pi^2} \geq \cdots$$

and this is clearly true. Also, we must check that

$$\lim_{n \to \infty} \frac{(2n+1)}{\pi^n} = 0$$

This is clearly true once we rewrite the limit in this form:

$$\lim_{n \to \infty} 2 \left( \frac{2}{\pi} \right)^n = 0$$

(reason: $2/\pi$ is less than 1, so raising this to a large power produces a small number). Because the two conditions to apply the test are satisfied, we have that, by Leibnitz test, the series is convergent.