MTH 142 Practice Problems for Exam 2 - Fall 2000

Last changed: October 31, 2000


Note: the exam will have fewer questions!

1. a) Obtain the first three nonzero terms of the Taylor series of \( f(x) = e^{-x^2} \) about \( a = 0 \).
   b) Use part (a) to calculate
   \[
   \lim_{x \to 0} \frac{1 + e^{-x^2}}{x^2}
   \]

2. a) Obtain \( P_3(x) \) = the Taylor polynomial of order 3 of \( \tan x \) about \( a = \pi/4 \)
   b) Use the plots of \( f(x) \) and \( P_3(x) \) to obtain an approximate value of the maximum error
   \( |f(x) - P_3(x)| \) for \( 0.7 \leq x \leq 1.2 \).

3. a) Calculate the radius of convergence of the series \( \sum_{n=0}^{\infty} 3^n (x - 2)^n \).
   b) Sketch in the number line an open interval of points \( x \) for which the series converges.

4. A solid \( S \) is produced by revolving about the x-axis the region \( R \) in the plane bounded by
   \( y = 1, \ y = 2x^2, \) and \( x = 0 \).
   a) Write down a Riemann sum that approximates the volume of the solid.
   b) Find the exact volume of the resulting solid.

5. Answer questions (a) and (b) of the previous problem, only now the solid is produced by
   revolving \( R \) about the y-axis.

6. An oil slick has the shape of a circle with radius 7,000 feet. After measurements were
   taken, it has been determined that the density of the oil at "r" feet from the center of the
   circle is given by the formula
   \[
   d(r) = \frac{0.004}{1 + r^2} \text{ Kg/ft}^2
   \]
   a) Write down a Riemann sum that approximates the total mass of the oil.
   b) Use part (a) to express the mass of the oil as an integral.

7. A 1 m. long rod has (linear) density \( d(x) = 2.0 + 0.015x \) gr/m, where \( x \) is the distance
   from one end.
   a) Obtain a Riemann sum that approximates the total mass of the rod.
   b) Calculate the total mass of the rod as an integral.
   c) Calculate the center of mass of the rod.
8. A 20 feet tall water tank has the shape of an inverted cone (i.e., the vertex is at the bottom) with circular top with radius 10 feet and height 20 feet. Water weighs 62.4 lbs/ft$^3$.
   a) Write down a Riemann sum that approximates the work required to take the water out of the tank from the top.
   b) Obtain the exact work in part (a) by calculating a suitable integral.

9. A dam has the shape of a trapezoid, with horizontal parallel sides measuring 30 ft. (bottom) and 60 ft (top). The height of the dam is 30 ft., and one vertical side is perpendicular to both base and top. The dam has water up to the top on one side. (Water weighs 62.4 lbs/ft$^3$.)
   a) Write down a Riemann sum that approximates the total force exerted by the water on the dam.
   b) Obtain an integral that gives the total force of the water on the dam.

10. A certain amount of fresh water shrimp is placed in a tank together with 2 lbs. of food, at 12:00 p.m. on January 1st. An additional 2 lbs. of food are added to the tank every 24 hours. After every 24 hours, 85% of the food either decomposes or is eaten. How much food is in the tank right before 12:00 p.m. on January 20th? Give details, and explain how you arrived at your answer.

11. If $P_2(x)$ is the order 2 Taylor polynomial approximating the function $f(x) = \frac{1}{2+x}$ near $x = 0$, estimate the error in the approximation $P_2(0.5) \approx f(0.5)$ by two methods:
   a) Using the plots of $P_2(x)$ and $f(x)$ directly.
   b) Using the inequality studied in class for error estimation when approximating a function by a Taylor polynomial.

**Determine if the series given below converges or diverges**
(Note: each one can be treated in more than one way!)

12. $\sum_{n=0}^{\infty} \frac{n^2}{1 + n^2}$

13. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{2^n}\right)$

14. $\sum_{n=3}^{\infty} \frac{(-2)^{n+1}}{\pi^n}$
SOLUTION MTH142 Practice Problems for Exam 2

1. a) Take \( y = -x^2 \) in \( e^y = 1 + \frac{1}{y} + \frac{1}{2!}y^2 + \cdots \) to get
\[
e^{-x^2} = 1 + \frac{1}{1}(-x^2) + \frac{1}{2}(-x^2)^2 + \cdots = 1 - x^2 + \frac{1}{2}x^4 + \cdots
\]
This gives the answer in less time than actually calculating the coefficients one by one (which is ok too).

b) Replace the function by the first few terms of the series to get
\[
\lim_{x \to 0} \frac{-1 + (1 - x^2 + \frac{1}{2}x^4)}{x^2} = \lim_{x \to 0} \frac{-x^2 + \frac{1}{2}x^4}{x^2} = \lim_{x \to 0} \frac{-1 + \frac{1}{2}x^2}{1} = -1
\]

2. a) Direct calculation gives the polynomial
\[
1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3
\]

b) From the plot we see that the gap between \( f(x) \) and \( p(x) \) is the largest possible at the endpoint \( x = 1.2 \). Therefore the maximum error in the interval \((0.7, 1.2)\) is \( D = f(1.2) - p(1.2) = 0.20911 \).

3. a) The radius of convergence is given by
\[
r = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \to \infty} \frac{n3^n}{(n+1)3^{n+1}} = \lim_{n \to \infty} \frac{n}{n+1} = \frac{1}{3}
\]

b) The interval \((2 - \frac{1}{3}, 2 + \frac{1}{3}) = (\frac{5}{3}, \frac{7}{3})\)

4. a) By taking sections perpendicular to the axis of rotation, we get “washers”. At the tickmark \( x_j \) the washer has inner radius \( r_j = 2x_j^2 \), outer radius \( R_j = 1 \), and thickness \( \Delta x \). The Riemann sum that approximates the volume is
\[
V \approx \sum_{j=0}^{n} (\pi R_j^2 - \pi r_j^2) \Delta x = \sum_{j=0}^{n} (\pi 1^2 - \pi (2x_j^2)^2) \Delta x
\]
The volume is obtained by taking limit as \( \Delta x \to 0 \). We have,
\[
Vol(S) = \int_{0}^{\sqrt{2}} (\pi - \pi 4x^4) \, dx = \frac{2\sqrt{2}\pi}{5} \approx 1.777153
\]

a) By taking sections perpendicular to the axis of rotation, we get “disks”. At the tickmark \( y_j \) the radius \( r_j = \frac{x_j}{\sqrt{y_j}} \) and thickness \( \Delta y \).

5. The volume is obtained by taking limit as \( \Delta y \to 0 \). We have,
The Riemann sum that approximates the volume is
\[
\sum_{j=0}^{n} \pi R_j^2 \Delta y = \sum_{j=0}^{n} \pi \left( \frac{\sqrt{y_j}}{2} \right)^2 \Delta y = \sum_{j=0}^{n} \pi y_j / 2 \Delta y
\]

b) The volume is obtained by taking limit as \( \Delta y \to 0 \). We have,
\[
Vol(S) = \int_{0}^{1} \frac{\pi}{2} y dy = \frac{\pi}{4}
\]

6. a) \[
\sum_{j=0}^{n} \frac{0.004}{1 + r_j^2} 2\pi r \Delta r
\]

b) \[
\int_{0}^{1000} \frac{(0.004) 2\pi r}{1 + r^2} \Delta r
\]

7. a) \[
\sum_{j=0}^{n} (2 + 0.015x) \Delta x
\]

b) \[
\int_{0}^{1} (2 + 0.015x) dx = 2.0075
\]

c) \[
\pi = \frac{\int_{0}^{1} x (2 + 0.015x) dx}{\int_{0}^{1} (2 + 0.015x) dx} = \frac{1.0050}{2.0075} = 0.50062266
\]

A cross-section of the cone (shown in the figure) is bounded by the lines \( y = \pm 2x \) and \( y = 20 \). Introduce tick marks in the \( y \)-axis.

8. The slab \( S_j \) at height \( y_j \) is a disk with radius \( R_j = x_j = y_j / 2 \) and thickness \( \Delta y \), so its volume is \( \pi (y_j / 2)^2 \Delta y \), and its weight is \( 62.4 \pi (y_j / 2)^2 \Delta y \). The work involved in raising the slab a distance of \( (20 - y_j) \) to the top of the cone is
\[
w_j = (20 - y_j) 62.4 \pi (y_j / 2)^2 \Delta y
\]

The total work is approximated by
\[
W \approx \sum_{j=0}^{n} (20 - y_j) 62.4 \pi (y_j / 2)^2 \Delta y
\]

The exact work is given by
\[
\int_{0}^{20} (20 - y) 62.4 \pi (y / 2)^2 \Delta y = 653451.2720
\]
9. a) A sketch of the dam is shown in the figure above. Note that the equation of the right hand, non-horizontal side is \( y = x - 30 \). Introduce tick marks \( y_0, y_1, \ldots, y_n \), in the \( y \) axis. At height \( y_j \), the slab has area \( (y_j + 30) \Delta y \), and the pressure at this height is \( 62.4 (30 - y_j) \). Therefore the force on the slab is

\[
F_j = 62.4 (30 - y_j) (y_j + 30) \Delta y
\]

The total force is approximated by

\[
F \approx \sum_{j=0}^{n} 62.4 (y_j + 30) (30 - y_j) \Delta y
\]

The exact value of the total force is obtained by taking the limit as \( \Delta y \to 0 \):

\[
F = \int_{0}^{30} 62.4 (y_j + 30) (30 - y_j) dy = 1,123,200
\]

10. The following table is helpful:

<table>
<thead>
<tr>
<th>day</th>
<th>amount before 24hrs are up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan 2</td>
<td>(0.15) 2</td>
</tr>
<tr>
<td>Jan 3</td>
<td>(0.15) 2 + (0.15)^2 2</td>
</tr>
<tr>
<td>Jan 4</td>
<td>(0.15) 2 + (0.15)^2 2 + (0.15)^3 2</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>Jan 20</td>
<td>(0.15) 2 + (0.15)^2 2 + \cdots + (0.15)^{19} 2</td>
</tr>
</tbody>
</table>

Then, the amount right before noon on January 20th is

\[
(0.15) 2 + (0.15)^2 2 + \cdots + (0.15)^{19} 2 = \frac{(0.15)2 (1 - (0.15)^{20})}{1 - 0.15} \approx 0.352941176
\]

11. a) The Taylor polynomial of degree 2 of \( f(x) \) is \( P(x) = \frac{1}{2} - \frac{1}{8} x + \frac{1}{3} x^2 \), so the absolute value of the error in approximating \( f(0.5) \) by \( P(0.5) \) is

\[
| f(0.5) - P(0.5) | = | 0.4 - 0.40625 | = 0.00625
\]

b) The function \( |f^{(3)}(t)| = | -6/(t+2)^4 | \) is decreasing on the interval \( (0, 0.5) \), so it attains its maximum in that interval at \( t = 0 \). The maximum is \( M = 6/2^4 = 3/8 \). The error bound when approximating \( f(0.5) \) by \( p(0.5) \) is

\[
|f(0.5) - p(0.5)| \leq M \frac{(0.5)^3}{3} = \frac{1}{128} \approx 0.00781
\]
12. Solution 1: Note that \(\frac{n^2}{1+n^2}\) behaves like \(\frac{1}{n}\) when \(n\) is large, so we suspect that the series diverges. The following inequalities are clearly valid:

\[
0 \leq \frac{n^2}{n^3 + n^3} \leq \frac{n^2}{1+n^3}, \quad n = 1, 2, \ldots
\]

The term in the center simplifies to \(\frac{1}{2n}\). Since \(\sum_{n=1}^{\infty} \frac{1}{2n}\) diverges, so does \(\sum_{n=1}^{\infty} \frac{n^2}{1+n^3}\). This is our original series with only the first term changed, so our original series \(\sum_{n=0}^{\infty} \frac{n^2}{1+n^n}\) also diverges.

Solution 2: We use the integral test. Set \(f(x) = \frac{x^2}{1+x^x}\) for \(x > 1\). Now \(\int_{1}^{\infty} f(x)dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x^2}{1+x^x}dx = \lim_{b \to \infty} \ln |1+x^3|_1^{b} - \frac{1}{3} \ln |1+b^3| - \frac{1}{3} \ln 2 = \infty\). Therefore, the series diverges too.

13. Solution 1: Note that \(\lim_{n \to \infty}(1 + \frac{1}{2^n}) = 1 \neq 0\). Therefore the series diverges.

Solution 2: Using comparison, we have

\[
0 \leq 1 \leq 1 + \frac{1}{2^n}, \quad n = 0, 1, \ldots
\]

Since \(\sum_{n=0}^{\infty} 1\) is divergent, so is our original series.

14. Solution 1:

\[
\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{\pi^n} = \sum_{n=0}^{\infty} -2 \left(\frac{-2}{\pi}\right)^n = \text{a convergent geometric series, since } |x| = \left|\frac{-2}{\pi}\right| < 1
\]

Solution 2: Since \(\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{\pi^n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{n+1}}{\pi^n}\) the series is of the alternating (sign) type. To apply Leibnitz test we must first check two things. The first is,

\[
\frac{2^1}{\pi^0} \geq \frac{2^2}{\pi^1} \geq \frac{2^3}{\pi^2} \geq \cdots
\]

and this is clearly true. Also, we must check that

\[
\lim_{n \to \infty} \frac{2^{n+1}}{\pi^n} = 0
\]

This is clearly true once we rewrite the limit in this form:

\[
\lim_{n \to \infty} 2 \left(\frac{2}{\pi}\right)^n = 0
\]

(reason: \(2/\pi\) is less than 1, so raising this to a large power produces a small number). Because the two conditions to apply the test are satisfied, we have that, by Leibnitz test, the series is convergent.