2 A point \((m, n)\) in the plane \(\mathbb{R}^2\) is a lattice point if both coordinates \(m\) and \(n\) are integers. Prove that the number of lattice points inside any circle centered at the origin is a number of the form \(4k + 1\) for some integer \(k\).

**Proof.** Let \(i \in \{1, 2, 3, 4\}\) and let \(Q_i\) be the set of lattice points in the \(i^{th}\) quadrant excluding \((0,0)\) and one of the axes. That is, \(Q_1 = \{(m, n) : m > 0, n \geq 0\}\), \(Q_2 = \{(m, n) : m \leq 0, n > 0\}\), \(Q_3 = \{(m, n) : m < 0, n \leq 0\}\), and \(Q_4 = \{(m, n) : m \geq 0, n < 0\}\).

Let \(C\) be the set of lattice points inside a circle centered at the origin. If \((m, n) \in Q_1 \cap C\), then \(m > 0\) and \(n \geq 0\), so \((-n, m) \in Q_2 \cap C\), \((-m, -n) \in Q_3 \cap C\), and \((n, -m) \in Q_4 \cap C\). It is clear that for \(i \neq j\), \(|C \cap Q_i| = |C \cap Q_j|\).

The set \(\{C \cap Q_1, C \cap Q_2, C \cap Q_3, C \cap Q_4, \{(0,0)\}\}\) is a partition of the set of \(C\). Thus, \(|C| = |C \cap Q_1| + |C \cap Q_2| + |C \cap Q_3| + |C \cap Q_4| + |\{(0,0)\}| = 4|C \cap Q_1| + 1\) \(\Box\)