**Theorem 21.** Brooks. *If* \( G \) *is a connected graph on* \( n \geq 3 \) *vertices, not an odd cycle and not a complete graph then* \( \chi(G) \leq \Delta(G) \).

**Proof:** Let \( k = \chi(G) \) and assume \( G \) is \( k \)-critical. Let \( d = \Delta(G) \).

The only connected graph with \( d = 1 \) is \( K_2 \), but we are not considering this case.

If \( d = 2 \), then \( G \) is an open or closed path and since \( G \) is not an odd cycle, we know that \( G \) can be two colored, hence \( k \leq d \).

The only connected graph with \( k = 1 \) is the complete graph \( K_1 \), and we are assuming that is not the case.

If \( k = 2 \) then \( G \) is bipartite and since \( G \) is 2-critical, removing any vertex \( v \) means we can color \( G - v \) with 1 color. So, \( G - v \) must be the null graph, so \( v \) is adjacent to all other vertices. Well if we remove one of those instead, then the remaining graph is the null graph. The only graph with these properties is \( K_2 \). But we are assuming \( G \) is not \( K_2 \).

If \( G \) is 3-critical, then removing any vertex gives a bipartite graph. So \( G - v \) has no odd cycles, for all \( v \in V(G) \), but \( G \) must contain an odd cycle, \( C \). Suppose \( G \) has some vertex \( u \) not on \( C \). Then \( G - u \) still contains the odd cycle and so is not 2-colorable. \( G \) must be an odd cycle.

Suppose \( G \) has a cut vertex, \( v \). Let \( C_1, C_2, \ldots, C_2 \) be the components of \( G - v \). Each \( C_i + v \) can be colored with \( k - 1 \) colors. Swap colors so that \( v \) is the same in each. Now we have a \( k - 1 \) coloring of \( G \). But this is a contradiction because \( G \) is \( k \)-critical.

Now we assume that \( k \geq 4 \), \( d \geq 3 \), and \( G \) is 2-connected.

Suppose we have a 2 element cut-set, \( \{u, v\} \).

Let \( C_1 \) be one of the components of \( G - \{u, v\} \) and let \( C_2 = G - (\{u, v\} \cup V(C_1)) \).
Let’s first suppose that $u$ and $v$ are adjacent.

We know that $G$ is $k$-critical, so $G - u$ can be colored with $k - 1$ colors. Let $d_1$ be the number of neighbors of $u$ in $C_1$ and let $d_2$ be the number of neighbors of $u$ in $C_2$. Both $d_1$ and $d_2$ are at most $d - 2$. If $k > d$ then $k - 3 \geq d - 2$, and we have 2 available colors $a_1, a_2$ for $u$ that are not the same as any of $u$’s neighbors in $C_1$ and we have 2 available colors $b_1, b_2$ for $u$ that are not the same as any of $u$’s neighbors in $C_2$. It could be that one of these colors in each case is the color used for $v$. Suppose that leaves color $a_1$ and color $b_1$. If they are not the same, we swap the colors of vertices which are colored either $a_1$ or $b_1$ in $C_1$ so that $b_1$ becomes free to use for $u$. Hence, $k \leq d$.

If $u$ and $v$ are not adjacent.

We know that $G$ is $k$-critical, so $G - u$ can be colored with $k - 1$ colors. Let $d_1$ be the number of neighbors of $u$ in $C_1$ and let $d_2$ be the number of neighbors of $u$ in $C_2$. Both $d_1$ and $d_2$ are at most $d - 1$. If $k > d$ then $k - 2 \geq d - 1$, and we have one available colors $a_1$ for $u$ that is not the same as any of $u$’s neighbors in $C_1$ and we have one available color $b_1$ for $u$ that is not the same as any of $u$’s neighbors in $C_2$. If they are not the same, we swap the colors of vertices which are colored either $a_1$ or $b_1$ in $C_1$ so that $b_1$ becomes free to use for $u$.

Finally, we assume $k \geq 4$, $d \geq 3$, and $G$ is 3-connected.

Since $G$ is connected and not the complete graph, there exists an induced $P_3 : u, v, w$ in $G$. Since $G - \{u, w\}$ is connected, we can form a search tree of $G$ with root vertex $v$. Thus we can order the vertices, $v_3, v_4, \ldots, v_n = v$ in such a way that each vertex $v_3, v_4, \ldots, v_{n-1}$ has a neighbor to the right of it in the list. For instance, you can first select all vertices in the highest level, then all in the next highest level, etc.
Let \( u = v_1, w = v_2 \) and follow the greedy algorithm to color the vertices of \( G \) with \( d \) colors.

\[ v_1, v_2, v_3, \ldots, v_{n-1}, v_n \]

Of course, \( u = v_1 \) and \( w = v_2 \) will get the same color, \( c_1 \). For each vertex, \( v_1, v_2, \ldots, v_{n-1} \), when it is encountered, it has at most \( d - 1 \) neighbors to the left, so that is the most number of colors that could have already been used on its neighbors. Thus we always have one of the \( d \) colors available to color the vertex. With the last vertex, \( v_n \), it may have \( d \) neighbors to the left, but at most only \( d - 1 \) colors will have been used since we know that two of its neighbors, \( u \) and \( w \) got the same color. \( \square \).

**Theorem 22.** Every simple graph with maximum degree \( \Delta \) has a proper \( \Delta + 1 \)-edge coloring.

**Proof:** By induction on \( q = |E(G)| \).

Let vertex \( u \) have degree \( \Delta(G) \) and remove some edge \( e = uw_1 \). By the induction hypothesis, we can color the remaining edges with \( \Delta + 1 \) colors. We will show that by possibly recoloring some edges, we can color \( uw_1 \) using no more than \( \Delta + 1 \) colors. Note that for each vertex there is at least one color that is not used on its incident edges.

Name the unused color at \( u \), color \( c_1 \). If \( c_1 \) is not used on an edge incident to \( v \) we are done, color the edge \( uw_1 \) with color \( c_1 \). But suppose \( c_1 \) is used at \( w_1 \) and color \( c_2 \) is not.

We can assume that \( c_2 \) is used on an edge, \( uw_2 \) incident to \( u \) or we could color \( uw_2 \) with color \( c_2 \). We recolor \( uw_1 \) by color \( c_2 \) and uncolor \( uw_2 \). The color \( c_1 \) must be used at \( w_2 \) or else we would be able to color \( uw_2 \) by color \( c_1 \). Look at the component of \( H(c_1, c_2) \) containing \( w_2 \). It must contain a \( w_2, v \)-path containing \( w_1 \) or else we could interchange colors \( c_1 \) and \( c_2 \) in that component and thus be able to color \( uw_2 \) by \( c_1 \).
Let $c_3$ be the unused color at $w_2$. Assume it is used at $v$ or else we could color $uw_2$ with color $c_3$. We color edge $uw_2$ with color $c_3$ and uncolor edge $uw_3$. It must be that $c_1$ is used on an edge incident to $w_3$ or we could color edge $uw_3$ with color $c_1$. Similarly to before, there must be a bi-colored path from $w_3$ to $v$ containing $w_2$ in $H(c_1, c_3)$ or else we would be able to switch colors $c_1$ and $c_3$ in the component containing $w_3$ allowing us to use color $c_1$ on edge $uw_3$.

At some point in this process, say at $w_k$ we may encounter a color, unused at $w_k$ that we have already named, $c_i$ where $i \leq k$.

If not then, we reach $w_\Delta$ with edge $uw_\Delta$ is colored with color $c_\Delta$ and unused color $c_{\Delta+1}$ at $w_\Delta$. There are no more edges incident to $u$, so we know that $c_{\Delta+1}$ is unused at $u$. At this point we color the uncolored edge $uw_{\Delta-1}$ with color $c_\Delta$ and use color $c_{\Delta+1}$ on $uw_\Delta$. See Figure 1.

So, if we had instead of encountering a new color $w_{k+1}$ unused at $w_k$, we encountered a color we have already named at $w_k$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
Suppose it is $c_i$ where $i \leq k$. We, as usual, use the color $c_k$ on edge $uw_{k-1}$ and uncolor the edge $uw_k$. We note that $c_1$ is used at $w_k$ or we would be able to color edge $uw_k$ with $c_1$ and be done with it. So consider the component of $H(c_1, c_i)$ that contains $w_k$. This component cannot be connected to the previously named component which contains a path from $w_i$ to $u$ through $w_{i-1}$. Because components of $H(c_1, c_i)$ must have only vertices of degree at most 2. We know that $u$ and $w_i$ have only degree 1, but we know that $u$ has only the edge colored $c_i$, namely $uw_{i-1}$ and no edge colored $c_1$ incident to it and $w_i$ has only the edge colored $c_1$ and no edge colored $c_i$ incident to it. Thus, we can swap the colors of $c_1$ and $c_i$ in the component of $H(c_1, c_i)$ that contains $w_k$ and use color $c_1$ on edge $uw_k$. □