MTH 316: ARE YOU READY?

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In mathematics you don't understand things. You just get used to them.

– John von Neumann

There is no getting away from it; MTH316 - Algebra is a **hard** course. The sheer amount of abstraction and generality of the results that we will discuss may well come as a shock. However, all is not lost! A well prepared student can easily do well in this class. My experience of teaching this course has indicated to me that one of the main problems is that many students are not at ease with many of the topics of the prerequisite courses MTH215 - Linear Algebra and MTH307 - Introduction to Mathematical Rigor. In order to partially remedy this, I have created this worksheet so that you know what topics from these courses you will be expected to know. I do not intend to review these in class (meaning we can spend time on some of the really interesting results!), and it will certainly make life easier for you when doing the multitude of exercises if you are not struggling with these more basic topics.

This document is merely designed for you to check your preparedness for the course. You should not consider it as a replacement for a good set of notes in the prerequisite courses MTH215 and MTH307.

1. MTH215 - Linear Algebra

1.1. **Matrix multiplication.** We will often have to carry out matrix multiplication (and addition, but this is usually well understood) in MTH316. However, if the matrix is 3×3 or bigger, you will often be given a formula for the computation. On the other hand, you really should be comfortable with addition and multiplication of 2×2 matrices. In particular recall that addition is simply carried out term-wise:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

On the other hand, multiplication is a little more complicated. Here is an example with numbers. Suppose we want to compute

$$\begin{pmatrix} 2 & -1 \\ \mathbf{3} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{-2} & 3 \end{pmatrix} = \begin{pmatrix} w & x \\ \mathbf{y} & z \end{pmatrix}$$

How do we compute the values of w, x, y and z? Remember that a matrix product is computed by multiplying *rows of the first matrix* with *columns of the second matrix* term-wise and then summing the results. For example, suppose we want to compute y in the matrix above. Then y is in the **second** row and **first** column of the target matrix. To get y therefore, we multiply the second row of the first matrix with the first column of the second matrix term by term (these are in bold above). Thus $y = (3 \times 1) + (1 \times (-2)) = 1$.

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Exercise. *Check that you agree with*

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 1 & 3 \end{pmatrix}.$$

The general formula for the product of two 2×2 matrices is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

It is important to realise that matrix multiplication is **not** commutative. That is, for general matrices A and B, it is not always true that $AB = BA$.

Exercise. *Verify the above by computing*

$$\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}.$$

You should also be aware (but will not need to prove) that matrix multiplication and addition are *associative* operations. That is, for matrices A, B and C, we have

- $A + (B + C) = (A + B) + C$ and
- $A(BC) = (AB)C$.

1.2. **Matrix determinants.** You need to be aware of some of the key properties of the determinant of a matrix. Recall that for the 2×2 matrix we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

The matrix also has some useful properties.

- A matrix M is invertible (see below) if $\det M \neq 0$.
- For matrices M and N we have $\det(MN) = \det M \det N$.

Knowledge of the determinant allows us to define two special families of square matrices.

- $\mathrm{GL}(n, \mathbb{R})$ is the *General linear group of degree n over* $\mathbb{R}$ which is defined as
$$\mathrm{GL}(n, \mathbb{R}) = \left\{ A \in \mathbb{R}^{n \times n} \,\middle|\, \det A \neq 0 \right\},$$
the set of $n \times n$ matrices with entries in $\mathbb{R}$ and which have non-zero determinant.
- $\mathrm{SL}(n, \mathbb{R})$ is the *Special linear group of degree n over* $\mathbb{R}$ which is defined as
$$\mathrm{SL}(n, \mathbb{R}) = \left\{ A \in \mathbb{R}^{n \times n} \,\middle|\, \det A = 1 \right\},$$
the set of $n \times n$ matrices with entries in $\mathbb{R}$ and which determinant equal to 1.

It should be obvious that $\mathrm{SL}(n, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$.

1.3. **Matrix inverses.** As with matrix multiplication, you will be given a formula for the inverse of a 3×3 or larger matrix. However, you should be able to compute the inverse A^{-1} of an invertible 2×2 matrix. Recall that the inverse A^{-1} of a matrix A is a matrix such that $AA^{-1} = A^{-1}A = I$, where I is the $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

which we call the *identity matrix*. In particular, for 2×2 matrices the identity matrix is $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$. From the section on determinants, we know that the inverse exists if and only if $\det A \neq 0$. Furthermore, simple computation tells us that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exercise. *Compute the inverses of*

$$\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

and check your answers by direct computation.

2. MTH307 - Mathematical Logic

2.1. **Proofs.** This course is very proof oriented. You should certainly be comfortable with the rudiments of logic, the language of mathematics (quantifiers, implications etc) and the standard proof techniques. Many of the homework exercises will require you to prove things, thus you are expected to write clear, correct proofs. A jumble of symbols will not be sufficient! As ever, the only way to learn proofs in this course will be by practice, practice and more practice. After a while, you will obtain an intuition as to what proof technique and arguments will work in a given situation.

2.2. **Relations.** It will be helpful to know the notion of a *relation* on a set A and the terms *reflexive*, *symmetric* and *transitive*. You should know what an *equivalence relation* is and how to prove a relation satisfies this condition. A relation R on a set A is

- reflexive if xRx for all $x \in A$.
- symmetric if xRy implies yRx for all $x, y \in A$.
- transitive if xRy and yRz implies xRz for all $x, y, z \in A$.
- an equivalence relation if it is reflexive, symmetric and transitive.

Recall that for an equivalence relation R, the equivalence class of A is defined as $[a] = \{x \in A \mid xRa\}$ and the equivalence classes form a *partition* of A.

Exercise. *Prove the following are equivalence relations and find the equivalence classes.*

(1) $\equiv \pmod{n}$ *on* $\mathbb{Z}$.
(2) R *on* $\mathbb{R} - \{0\}$ *defined by* xRy *if and only if* $xy > 0$.
(3) R *on* $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ *defined by* $(m, n)R(p, q)$ *if and only if* $mq = np$.

In the first example, the set of equivalence classes is denoted by $\mathbb{Z}_n$. *What common set do the equivalence classes in the third example "represent"?*

The first example above, giving the set $\mathbb{Z}_n$ will be very important, and will appear regularly throughout the course. You should be adept at carrying out the standard arithmetic operations (addition and multiplication) on $\mathbb{Z}_n$, which is essentially all carried out modulo n. If you have taken the number theory course, you may well have gained more experience with these objects.

2.3. **Functions.** We will make use of the terms *injection* (one-to-one), *surjection* (onto) and *bijection*. You should know what these mean, and also be able to prove that a given function does or does not satisfy these properties. Let $f : A \to B$ be a function.

- We say f is an injection if $f(x) = f(y)$ implies $x = y$. Informally, everything in B is mapped to by at most one element of A.

- We say f is a surjection if for all $y \in B$ there exists $a \in A$ such that $f(a) = b$. Informally, everything in B is mapped to at least once by f.
- We say f is a bijection if it is both an injection and a surjection. Informally, everything in B is mapped to exactly once under f.

Exercise. *Prove the following maps are bijections.*

(1) $f : \mathbb{R} \to (0, \infty)$ *defined by* $f(x) = 2^x$.
(2) $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x^3$.
(3) $f : \mathbb{Z}_4 \to \mathbb{Z}_5 - \{0\}$ *defined term-wise by* $f(0) = 1$, $f(1) = 2$, $f(2) = 4$ *and* $f(3) = 3$.

In the final case, check that f *also satisfies* $f(x + y) = f(x) \times f(y)$, *where addition is carried out in* $\mathbb{Z}_4$ *and multiplication in* $\mathbb{Z}_5$.

Recall that the composition of the functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$. Note the ordering here: we read from right to left, so that we first apply f and then g. You should also know that functional composition is associative. That is, for functions f, g and h, we have $f \circ (g \circ h) = (f \circ g) \circ h$. However, this composition is not commutative, since if $f(x) = x^2$ and $g(x) = x + 1$ then $f \circ g(x) = (x + 1)^2 = x^2 + 2x + 1$ but $g \circ f(x) = x^2 + 1$. Furthermore, if f and g are both injections/surjections/bijections then the composition $g \circ f$ is also an injection/surjection/bijection.

2.4. **Complex Numbers.** We will make use of the complex numbers at certain times in the course. You should know the definition of a complex number $z = a+bi$, or the radial form $z = r(\cos\theta + i \sin\theta)$ where $i = \sqrt{-1}$. You should be able to carry out standard computations of complex numbers (i.e addition, multiplication and computing the absolute value $|z|$).

3. Attitude

The most important thing you can bring to this course is a willingness to interact with abstract mathematical ideas and appreciate how they allow us to prove very general results. There is no substitute in mathematics for practice - the more you interact with a concept, the more you will understand it and become comfortable with it. It is also helpful to build up a catalogue of examples, so that when you first come across a new definition or theorem, you can check it against something you have already encountered. Most of this course will simply be generalising concepts you are extremely comfortable with (addition on $\mathbb{Z}$, multiplication on $\mathbb{R} - \{0\}$, addition on $\mathbb{Z}_n$, isometries of the square, matrix multiplication), so try to use these as your intuition throughout the course.