INTRODUCTION TO MTH 316

> *Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.'*
>
> – Sir Michael Atiyah

This document is designed to help familiarise you with some of the concepts we will cover in MTH 316 - Algebra. We will be discussing the problems on this sheet in groups (no pun intended) during the first class of the semester. Try to prepare for the class by attempting these practice problems. A good way to build an intuition with groups is to consider connections with geometry. There are some examples of this within this document.

1. Binary operations

Let G be a set. A *binary operation* on G is a function $*\colon G \times G \to G$. That is, $*$ takes *any* two elements of G and combines them to produce another element of G. We write

$$*(g_1, g_2) = g_1 * g_2 = g_1 g_2.$$

Since the output belongs to the same set as the two input elements, we say that G is *closed* under the operation $*$.

Example. You will be familiar with a number of binary operations. Some examples are:

- $+$ on $\mathbb{Z}$.
- $+$ on $\mathbb{Z}_4$.
- $\cup$ on $\mathscr{P}(X)$ for some set X.
- $\div$ on $\mathbb{Q} - \{0\}$.
- $-$ on $\mathbb{Z}$.
- Multiplication on $\mathrm{GL}(2, \mathbb{R})$[1].
- $\circ$ (compositon of functions) on the set of bijections $\beta\colon X \to X$ for some set X (these functions are called permutations of the set X).
- . . . and many more. . .

Note that for example we are used to writing $a + b$ instead of $+(a, b)$. We will continue this convention. Binary operations are a fundamental concept in algebra, as you can see from the examples above.

The next exercise includes some more binary operations.

Problem 1. *Which of the following are binary operations on the given set? You need to check that the binary operation works for any pair of inputs, and that it is closed.*

(1) $\times$ *on* $\mathbb{R} - \mathbb{Q}$.
(2) $\times$ *on* $\mathbb{Q} - \{0\}$.
(3) $\circ$ *on the set of functions* $f\colon X \to X$ *for some set* X.

[1] See the "Are you Ready?" worksheet

(4) $|x-y|$ *on the set* $\mathbb{R}$.

We say that a binary operation $*$ is associative if for all a, b, c, we have $a*(b*c) = (a*b)*c$. Again, we know from previous courses that a lot of binary operations are associative.

Example.

- $+$ on $\mathbb{Z}$ (or any set).
- $\times$ on $\mathbb{R}$.

Problem 2. *For the closed binary operations in the previous question, check whether or not they are associative.*

If the set G is finite, we are able to fully write out the operation table for $*$ on G. For example, we can write the table for addition on $\mathbb{Z}_4$ as follows.

$\mathbb{Z}_4, +$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Problem 3. *Now write out the operation tables for the following four element sets.*

- $\times$ (mod 8) *on* $\{1,3,5,7\}$.

$\times$	1	3	5	7
1				
3				
5				
7				

- $\times$ *on* $\{1,-1,i,-i\}$.

$\times$	1	-1	i	$-i$
1				
-1				
i				
$-i$				

- $\times$ *on* $\mathbb{Z}_5 - \{0\}$.

$\times$	1	2	3	4
1				
2				
3				
4				

- *Matrix multiplication on the set*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\times$	I	A	B	C
I				
A				
B				
C				

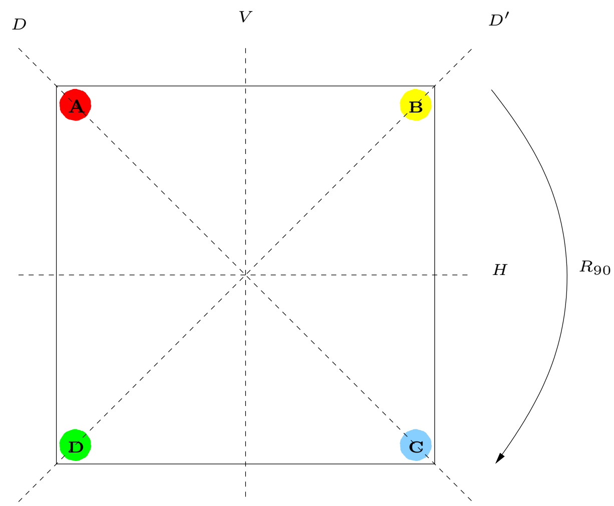
Make a note of any observations you make.

Before we give the definition of a group (all the above are examples), we will construct another example using our geometric intuition. In fact, there is a very close relationship between algebra and geometry (there is in fact an entire subject called *Algebraic Geometry*). You have already seen this in e.g MTH243 and MTH 215 - vector products and determinants have both algebraic and geometric descriptions. Suppose we are given a square, and want to consider all function which map the square onto itself while preserving distances. Those of you who have taken geometry will know this is called an *isometry* of the square.

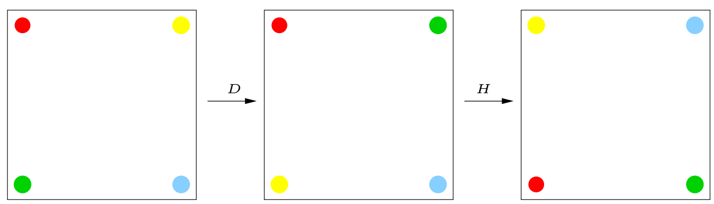
We can convince ourselves that an isometry of the square must map a corner to another corner. One obvious way of doing this is to rotate the square around its centre by $90°$; we call this operation R_{90}. Of course, we can also rotate through $180°$ and $270°$, giving the operations R_{180} and R_{270}, but when we rotate one more time, through $360°$, we see that this is the same as rotating through $0°$ - in other words, not rotating at all! We denote this final operation by R_0. Another way of mapping corners to corners is reflecting the square through a line which passes through its centre. There are four such reflections - we label them H, V, D and D' (horizontal, vertical and diagonals) corresponding to the lines in the diagram below. Clearly, applying a reflection twice returns us back to the original position (so is equivalent to rotation through $0°$). Note that the set $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$ gives the complete list[2] of isometries of the square.

Here comes the crucial observation. Suppose we apply any two of the above isometries (by composing them as functions). Then the result of this composition will map corners of the square to corners, and thus will be another isometry of the square. Thus this composition of isometries is a binary operation on D_4.

[2] The notation will be explained during the semester.



For example, suppose we carry out R_{90} followed by R_{180}. It is easy to see that the final result is equivalent to just applying R_{270}; thus we may write $R_{180}R_{90} = R_{270}$ (we use functional composition here, so we should read it from "right to left"). Here is a harder example: What is HD? It is probably easiest to "get our hands dirty" and just do the calculation.



Looking at this final picture, we can see this end result is the same as we would obtain by carrying out just R_{270}. Thus we may write $HD = R_{270}$.

Problem 4. *As we did above, we can write out an operation table for this collection of functions. Does this table have any similar properties to those you observed above? Are there any differences?*

$\times$	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0								
R_{90}			R_{270}					
R_{180}								
R_{270}								
H								
V								
D						R_{270}		
D'								

If a binary operation $*$ satisfies $a * b = b * a$ for all a, b, we say it is *commutative*. Hopefully the observations you have made will arise in the following definition, which is the fundamental object of study in MTH316.

Definition. A **group** is a set G along with a binary operation $*$ which satisfies the following properties.

- (Closure) The operation $*$ is closed, so that $a * b \in G$ for all $a, b \in G$..
- (Associativity) The operation $*$ is associative, so that $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.
- (Identity) There exists an identity element $e \in G$ such that for all $g \in G$ we have $e * g = g * e = g$.
- (Inverses) For all elements $g \in G$, there exists an inverse to g. That is, for each $g \in G$, there exists an element h (which depends on the choice of g) such that $g * h = h * g = e$.

Notice that in particular, we don't demand that the binary operation $*$ is commutative (much to the disappointment of many a student!) When thinking about groups, you should have some "working examples" in your mind. A good place to start are the exampes you computed above. When you come across a theorem, see if it works with these.