INTRODUCTION TO MTH 316

This document is designed to help familiarise you with some of the concepts we will cover in MTH 316 - Algebra. We will be discussing the problems on this sheet in groups (no pun intended) during the first class of the semester. Try to prepare for the class by attempting these practice problems. A good way to build an intuition with groups is to consider connections with geometry. There are some examples of this within this document.

1. Binary opeations

Let G be a set. A *binary operation* on G is a function $*\colon G\times G\to G$. That is, $*$ takes *any* two elements of G and combines them to produce another element of G. We write

$$*(g_1,g_2)=g_1*g_2=g_1g_2.$$

Since the output belongs to the same set as the two input elements, we say that G is *closed* under the operation $*$.

Problem 1. *Show that the following are binary operations on the given set. You need to check that the binary operation works for any pair of inputs, and that it is closed.*

(1) $+$ *on* $\mathbb{Z}$.
(2) $+$ *on* $\mathbb{Z}_4$.
(3) $\times$ *on* $\mathbb{R}-\{0\}$.
(4) $\times$ *on* $\mathbb{Q}$.
(5) $\times$ *on* $(0,\infty)$
(6) *Addition on* 2×2 *matrices,* $\mathrm{GL}(2,\mathbb{R})$.
(7) *Multiplication on* $\mathrm{GL}(2,\mathbb{R})$.
(8) *Multiplication on* 2×2 *matrices with determinant* 1*,* $\mathrm{SL}(2,\mathbb{R})$.
(9) $-$ *on* $\mathbb{Z}$.
(10) $\div$ *on* $\mathbb{Q}-\{0\}$.
(11) $\cup$ *on* $\mathscr{P}(\{1,2,3\})$[1]
(12) $\cap$ *on* $\mathscr{P}(\{1,2,3\})$
(13) $\circ$ *(compositon of functions) on the set of bijections from* $\{1,2,3\}$ *to itself (these functions are called permutations of the set* $\{1,2,3\}$*).*
(14) *(tricky)* $+$ *on the set of functions* $f\colon\mathbb{R}\to\mathbb{R}$.
(15) *(tricky)* $+$ *on the set of function* $f\colon[0,1]\to[0,1]$.
(16) *(hard)* $\circ$ *(as above) on* D_8*, the set of isometries of the square. (We will discuss this one in class)*

We say that a binary operation $*$ is associative if for all a,b,c, we have $a*(b*c)=(a*b)*c$. For example, addition is associative because it is always true that $a+(b+c)=(a+b)+c$.

Problem 2. *For the binary operations in the previous question, check whether or not they are associative.*

[1] Recall that the power set $\mathscr{P}(X)$ of a set X is the set of subsets of X.

If the set G is finite, we are able to fully write out the operation table for $*$ on G. For example, we can write the table for addition on $\mathbb{Z}_4$ as follows.

$\mathbb{Z}_4, +$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Observation. What do you notice about the table? Here are some ideas to help you.

- What do you notice about each row and column?
- If a is any number, what can you say about $a + 0$ and $0 + a$? Are there any other numbers which has the same property as 0?
- If a is any number, are we able to find b so that $a + b = 0 = b + a$?

Problem 3. *Now write out the operation tables for the following.*

- $+$ *on* $\mathbb{Z}_3$.
- $\times$ *on* $\mathbb{Z}_2$.
- $\times$ *on* $\mathbb{Z}_5 - \{0\}$.

Do these tables all have the same properties as $\mathbb{Z}_4$. *Do some of them? Is there an odd one out?*

If $*$ is a binary operation on G, then we say the element $e \in G$ is the *identity* if $eg = ge = g$ for all $g \in G$. For example, 1 is an identity for multiplication on $\mathbb{Q}$ since for any rational number x we have $x \times 1 = 1 \times x = x$.

Problem 4. *How many identities can a binary operation have? To help you, consider a set* $A = \{a, e_1, e_2\}$ *where both* e_1 *and* e_2 *are identities. Write out a multiplication table for* A *satisfying this property. What goes wrong?*

- *Can you prove the identity is always unique for a binary operation? (We will do this in class, but it is good practice to try it now). Use the example above to help you.*

Suppose $*$ is a binary operation on a set G with identity element e. If a is any element of G, we say that b is an inverse to a if $a * b = b * a = e$. For example, given the rational number $\frac{3}{2}$, the inverse of $\frac{3}{2}$ under multiplication is $\frac{2}{3}$ since $\frac{3}{2} \times \frac{2}{3} = \frac{2}{3} \times \frac{3}{2} = 1$, and as we saw above, 1 is the multiplicative idntity for multiplication on $\mathbb{Q}$.

Problem 5. *For all the associative operations in Problem 2, decide if there is an identity element and if every element has an inverse. Keep a note of all those that do.*

The main object of study in this course will be groups. We have seen a number of these already.

Definition. A group is a set G along with a binary operation $*$ which satisfies the following properties.

- (Closure) The operation $*$ is closed.
- (Associativity) The operation $*$ is associative.
- (Identity) There exists an identity element $e \in G$ such that for all $g \in G$ we have $e * g = g * e = g$.
- (Inverses) For all elements $g \in G$, there exists an inverse to g. That is, for each $g \in G$, there exists an element h (which depends on the choice of g) such that $g * h = h * g = e$.

Notice that all the sets you recorded in Problem 5 are examples of groups! Have we seen any others?

2. Groups

Problem 6. *Now consider the set of rotations of the square. Let* R *be the rotation through* 90 *degrees,* R^2 *the rotation through* 180 *degrees and so on. Is this a group? If so, write out the group table for the set of rotations on a square. Have you seen this table before?*

Problem 7. *This time, we consider the set of reflections on the square. Let* H *be the reflection through the horizontal axis of symmetry,* V *the reflection through the vertical axis of symmetry,* D *the reflection across the diagonal from the top left to the bottom right of the square and* D' *the reflection across the diagonal from the top right to the bottom left of the square.*

- *Is the set of reflections of the square closed under composition?*
- *Is it true that the composition is commutative - e.g, if* F_1 *and* F_2 *are reflections then must we have* $F_1F_2 = F_2F_1$*?*

Problem 8. *The set* D_4 *of* isometries *on the square is made up of all the rotations and reflections of the square.*

- *Write out the group table for* D_4*.*
- *Notice that the set of rotations of the square are both a* subset *of the group* D_4 *and also a* group *in its own right. Can you find any other subsets of* D_4 *which are also groups in their own right?*
- *(hard) What do you notice about the size of the subsets that are themselves groups?*