MTH307 - HOMEWORK 9

Solutions to the questions in Section B should be submitted by the start of class on 12/06/18.

A. Warm-up Questions

Question A.1. For the following relations, decide if they are reflexive, symmetric or transitive, and if they are equivalence relations.

(i) $\leq$ on $\mathbb{R}$.
(ii) $\neq$ on $\mathbb{Z}$.
(iii) $\sim$ on $\mathbb{R}^2$ given by $(x,y) \sim (u,v)$ if and only if $x^2+y^2 = u^2+v^2$.
(iv) $\sim$ on $\mathbb{R}$ given by $x \sim y$ if and only if $|x-y| < 1$.
(v) $\sim$ on $\mathbb{Z}$ given by $x \sim y$ if and only if $|x-y| < 1$.
(vi) $|$ (divides) on the set $\{1,2,3,4,5,6\}$.
(vii) $\subseteq$ on the set $\mathcal{P}(\mathbb{R})$.
(viii) $\sim$ on $\mathbb{Z}$ given by $x \sim y$ if and only if $5|(x-y)$.
(ix) $\sim$ on $\mathbb{R}$ given by $x \sim y$ if and only if $xy \geq 0$.
(x) $\sim$ on $\mathbb{R}$ given by $x \sim y$ if and only if $xy > 0$.

Question A.2. Prove that the following relations are equivalence relations, and then compute the given equivalence classes.

(i) $\sim$ on $\mathbb{R}$ defined by $x \sim y$ if and only if $x^2 = y^2$. Find $[0], [2]$ and $[-3]$.
(ii) $\sim$ on $\mathbb{R}^2$ by $(x,y) \sim (u,v)$ if and only if $x^2v = u^2y$. Find $[(1,2)], [(0,0)]$ and $[(3,-1)]$.
(iii) $\sim$ on $\mathbb{R}$ by $n \sim m$ if and only if $n-m \in \mathbb{Z}$. Find $[3], [0.5]$ and $[\pi]$.
(iv) $\sim$ on $\mathbb{Z}$ by $n \sim m$ if and only if $3 \mid (n-m)$. Find $[0], [1]$ and $[2]$.
(v) $\sim$ on $\mathcal{P}(\mathbb{N})$ by $X \sim Y$ if and only if there exists a bijection $f\colon X \to Y$. Find $[\varnothing]$, $[\{3\}]$ and $[\mathbb{N}]$. (This one is a little harder than the others)

Question A.3. Consider the set $A = \{1,2,3,4\}$. Give a relation $R \subseteq A \times A$ satisfying the following.

(i) R is reflexive, symmetric and transitive.
(ii) R is reflexive, symmetric and not transitive.
(iii) R is reflexive, not symmetric and transitive.
(iv) R is reflexive, not symmetric and not transitive.
(v) R is not reflexive, symmetric and transitive.
(vi) R is not reflexive, symmetric and not transitive.
(vii) R is not reflexive, not symmetric and transitive.
(viii) R is not reflexive, not symmetric and not transitive.

B. Submitted Questions

Question B.1. Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x-y \in \mathbb{Q}$. Prove that $\sim$ is an equivalence relation and describe the equivalence class $[17]$.

Question B.2. Suppose $\sim$ is a relation on a non-empty set X with the property that for all $x \in X$ there exists $y \in X$ such that $x \sim y$. Suppose $\sim$ is symmetric and transitive. Prove $\sim$ is an equivalence relation.

C. Challenge Questions

Question C.1. We show a way of using equivalence relations to construct the rationals from the integers. Define $Q = \mathbb{Z} \times \mathbb{N}$ and define $\sim$ on Q by

$$(a,b) \sim (c,d) \text{ if and only if } ad = bc.$$

Show that $\sim$ is an equivalence relation. We now denote $\mathbb{Q} = Q/\sim$. Define

$$[(a,b)] + [(c,d)] = [(ad+bc, bd)] \text{ and } [(a,b)] \cdot [(c,d)] = [(ac,bd)]$$

(i) Show that these operations are well defined - they do not depend on the representative chosen.
(ii) Show that for all $[(a,b)] \in \mathbb{Q}$ we have $[(a,b)] + [(0,1)] = [(a,b)]$.
(iii) Show that for all $[(a,b)] \in \mathbb{Q}$ we have $[(a,b)] \cdot [(1,1)] = [(a,b)]$.