MTH307 - HOMEWORK 8

Solutions to the questions in Section B should be submitted by the start of class on 11/15/18.

A. Warm-up Questions

Question A.1.

(i) Let X be a set. Prove that $I_X\colon X \to X$, given by $I_X(x) = x$ is a bijection.
(ii) Show that any function $f\colon X \to \operatorname{ran} f$ is a surjection.

Question A.2. Decide if the following functions are injective and/or surjective, and prove your answer. If they are bijections, compute the inverse function.

(i) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$.
(ii) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 2$.
(iii) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x - |x|$.
(iv) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x + y$.
(v) $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (x - y, y + z)$.
(vi) $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$, $f(n) = \{1, 2, \ldots, n\}$.
(vii) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (y - x^2, x)$.
(viii) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x, y, x^2 + y^2)$.

Question A.3. Let $f\colon X \to Y$ be a function. We say $g\colon Y \to X$ is a *left inverse* of f if $g \circ f = I_X$ and we say $g\colon Y \to X$ is a *right inverse* of f if $f \circ g = I_Y$.

(i) Prove that f has a left inverse if and only if it is injective.
(ii) Prove that f has a right inverse if and only if it is surjective.
(iii) If a function has a left inverse, must the left inverse be unique? What about a right inverse?

Question A.4. Let X, Y and Z be sets.

(i) Show that $f\colon X \to Y$ is injective if and only if for all $h_1\colon Z \to X$ and $h_2\colon Z \to X$, $f \circ h_1 = f \circ h_2$ implies $h_1 = h_2$.
(ii) Show that $f\colon X \to Y$ is surjective if and only if for all $h_1\colon Y \to Z$ and $h_2\colon Y \to Z$, $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$.

B. Submitted Questions

Question B.1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (y^3, xy)$. Decide if f is injective and/or surjective and prove your answer.

Question B.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} -x & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$

(i) Prove that f is a bijection.
(ii) Find f^{-1}.

C. Challenge Questions

Question C.1. Prove the following.

(i) $f\colon \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{-}{n+1}2 & \text{if } n \text{ is odd.} \end{cases}$$

is a bijection. What is the inverse?
(ii) $f\colon \mathbb{Q} \cap (0, \infty) \to \mathbb{N}$ given by $f\left(\frac{m}{n}\right) = 2^m 3^n$ (where $\frac{m}{n}$ is fully reduced) is an injection.

Question C.2. Let X be a set with m elements and Y a set with n elements.

(i) If $m \leq n$, how many injections $f\colon X \to Y$ are there?
(ii) If $m \geq n$, how many surjections $f\colon X \to Y$ are there?
(iii) If $m = n$, how many bijections $f\colon X \to Y$ are there?