MTH307 - HOMEWORK 7

Solutions to the questions in Section B should be submitted by the start of class on 11/08/18.

A. Warm-up Questions

Question A.1. Compute dom f and ran f for the following functions (the natural domain is $\mathbb{R}$ in each case) and prove your answers.

(i) $f(x) = 3 - 4x$.
(ii) $f(x) = x^2 + 3$
(iii) $f(x) = x^2 + 6x - 4$.
(iv) $f(x) = x^2 - x - 1$.
(v) $f(x) = \frac{x-1}{x+1}$.
(vi) $f(x) = 3 - \sqrt{2+x}$.
(vii) $f(x) = \frac{3x-5}{2x-1}$.
(viii) $f(x) = \frac{4}{1+x^4}$.

Question A.2. Compute ran f for the following functions and prove your answer.

(i) $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, $f(m,n) = m + n$.
(ii) $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, $f(m,n) = 2m + n + 1$.
(iii) $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, $f(m,n) = 2m + 4n$.
(iv) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = y$.
(v) $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} n+4 & \text{if } n \text{ is even,} \\ n-2 & \text{if } n \text{ is odd.} \end{cases}$$

(vi) $f\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $f(X) = \mathbb{R} - X$.

Question A.3. Compute $g \circ f$ and $f \circ g$ for each of the following functions.

(i) $f, g\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$, $g(x) = x - 1$.
(ii) $f, g\colon \mathbb{R} - \{1\} \to \mathbb{R}$, $f(x) = g(x) = \frac{x}{x-1}$.
(iii) $f\colon \mathbb{R} - \{-2\} \to \mathbb{R}$, $g\colon \mathbb{R} - \left\{\frac{3}{5}\right\}$, $f(x) = \frac{3x+5}{x+2}$ and $g(x) = \frac{2x-1}{3-5x}$.
(iv) $f, g\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$, $g(x) = x^2 - 1$.

B. Submitted Questions

Question B.1. Find ran f for the following functions.

(i) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (y - x^2, x)$.
(ii) $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} n+4 & \text{if } n \text{ is even,} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}$$

C. Challenge Questions

Question C.1. *A formal definition of a function.* We give a set-theoretic definition of a function $f\colon X \to Y$. We says that $f \subseteq X \times Y$ is a function $f\colon X \to Y$ if for each $x \in X$ there exists exactly one $y \in Y$ such that $(x,y) \in f$. In this case we write $y = f(x)$.

(i) Show that for the above definition, we have $f = G_f$, the graph of f.
(ii) Show that, given a function f, we may compute dom $f = \{x \mid (\exists y)(x,y) \in f\}$.
(iii) Show that, if we take this definition of a function, then two functions can be equal even if they have different codomains (but the ranges must be equal).

Question C.2. We say that $d\colon \mathbb{R}^2 \to \mathbb{R}$ is a *metric* if it satisfies the following properties for all $x, y, z \in \mathbb{R}$.

(1) $d(x,y) \geq 0$.
(2) $d(x,y) = 0$ if and only if $x = y$.
(3) $d(x,y) = d(y,x)$.
(4) $d(x,y) \leq d(x,z) + d(z,y)$.

Prove the following are metrics on $\mathbb{R}$.

(i) $d(x,y) = |x - y|$.
(ii) $d(x,y) = 1$ for all $x \neq y$ and $d(x,x) = 0$ for all $x \in \mathbb{R}$.
(iii) $d(x,y) = \frac{|x-y|}{1+|x-y|}$.