MTH307 - HOMEWORK 6

Solutions to the questions in Section B should be submitted by the start of class on 11/1/18.

A. Warm-up Questions

Question A.1. Let $A = \{a, b\}$ and $B = \{a, c, d\}$. Compute the following.

(a) $\mathcal{P}(A)$.
(b) $\mathcal{P}(\mathcal{P}(A))$.
(c) $\mathcal{P}(A) - \mathcal{P}(B)$.
(d) $\mathcal{P}(A - B)$.
(e) $\mathcal{P}(A \cup B)$.
(f) $\mathcal{P}(A) \cup \mathcal{P}(B)$.
(g) $\mathcal{P}(A \cup B)$.
(h) $\mathcal{P}(A \cap B)$.
(i) $\mathcal{P}(A) \cap \mathcal{P}(B)$.
(j) $A \times \mathcal{P}(\mathcal{P}(\varnothing))$.

Question A.2. Let A and B be sets. Prove or disprove the following.

(a) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.
(b) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
(c) $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

Question A.3. For the following sets A_i, compute $\bigcap_{i \in \mathbb{N}} A_i$ and $\bigcup_{i \in \mathbb{N}} A_i$ and prove your answers.

(a) $A_i = (-i, i)$.
(b) $A_i = \left(0, \frac{1}{i}\right)$.
(c) $A_i = \left[\frac{1}{i}, i\right]$.
(d) $A_i = \left(\frac{1}{i}, i\right)$.
(e) $A_i = \left[2 - \frac{1}{i}, 5 + \frac{1}{i}\right]$.

Question A.4.

(a) For each $r \in \mathbb{R}$ define $\ell_r = \{(x, rx) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Compute $\bigcap_{r \in \mathbb{R}} \ell_r$ and prove your answer.
(b) For $x \in [0, 1]$ define $A_x = [0, x] \times [\sqrt{x}, 1]$. Compute $\bigcap_{x \in [0,1]} A_x$ and $\bigcup_{x \in [0,1]} A_x$ and prove your answers.

Question A.5. For $\alpha \in \mathbb{R}$, define

$$G_\alpha = \{(x, \alpha(x^2 - 1)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}.$$

(a) Sketch (on the same diagram) the sets G_0, G_1 and G_{-1}.
(b) Prove that $\bigcap_{\alpha \in \mathbb{R}} G_\alpha = \{(1, 0), (0, 1)\}$.

B. Submitted Questions

Question B.1. Let A and B be sets. Prove or disprove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Question B.2. For $\alpha \in \mathbb{R}$ define

$$G_\alpha = \{(x, e^{\alpha x}) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}.$$

Find $\bigcap_{\alpha \in \mathbb{R}} G_\alpha$ and prove your answer.

C. Challenge Questions

Question C.1. Let X be a non-empty set. Compute the following.

(a) $\bigcap_{A \in \mathcal{P}(X)} A$.
(b) $\bigcup_{A \in \mathcal{P}(X)} A$.

Question C.2. Let X be any set and suppose $A, B \in \mathcal{P}(X)$. Recall that the symmetric difference is $A \triangle B = (A - B) \cup (B - A)$.

(a) Prove that $A \triangle B = B \triangle A$.
(b) Show that there exists a unique $E \in \mathcal{P}(X)$ such that $A \triangle E = A$ for all $A \in \mathcal{P}(X)$.
(c) Show that for all $A \in \mathcal{P}(X)$, there exists $B \in \mathcal{P}(X)$ such that $A \triangle B = E$.
(d) Show that if $A, B \in \mathcal{P}(X)$ then there exists a unique $C \in \mathcal{P}X$ such that $A \triangle C = B$.