MTH307 - HOMEWORK 4

Solutions to the questions in Section B should be submitted by the start of class on 10/11/18.

A. Warm-up Questions

Question A.1. Prove the following by induction.

(a) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
(b) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.
(c) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n}(2i-1) = n^2$.
(d) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$.
(e) For all $n \geq 5$ we have $n^2 < 2^n$.
(f) For all $n \geq 10$, we have $n^3 < 2^n$.
(g) For all $n \in \mathbb{N}$, we have $8 \mid (3^{2n} - 1)$.
(h) For all $n \in \mathbb{N}$, we have $3 \mid (n^2 + 5n + 6)$.

Question A.2. Recall that the *Fibonacci sequence* is defined recursively by $f_1 = 1$, $F_2 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Prove the following for all $n \in \mathbb{N}$.

(a) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.
(b) $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$.
(c) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.
(d) $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$.

Question A.3.

(a) Suppose $a_1 = 1$, $a_2 = 2$ and $a_n = 10a_{n-1} - 7a_{n-2}$ for all $n \geq 3$. Prove that $a_n = 2^{n-1}$ for all $n \in \mathbb{N}$.
(b) Suppose $a_1 = 1$, $a_2 = 2$ and $a_n = 2a_{n-1} - a_{n-2}$. Find a closed formula for a_n and prove it.
(c) Suppose $a_1 = 2$ and $a_2 = 6$ and that $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 3$. Prove that $a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$ for all $n \in \mathbb{N}$.

Question A.4. Criticise the following proof by induction.
Proposition. All cows are the same colour.
Proof. We show that for all $n \in \mathbb{N}$, all collections of n cows are the same colour. First note that if $n = 1$, this is clear, since all collections of 1 cow have a common colour. For the inductive step, assume that for some $k \geq 1$, all collections of k cows are the same colour. Now consider a collection of $k+1$ cows. Label the cows $c_1, c_2, \ldots, c_{k+1}$. Then notice that $c_1, \ldots, c_k$ is a collection of k cows so they all are the same colour. But then $c_2, \ldots, c_{k+1}$ is a collection of k cows, so they also must be the same colour. Since c_2 belongs to both of these collections, all the $k+1$ cows must be the same colour. Hence by induction, if $n \in \mathbb{N}$ then all collections of k cows are the same colour, and so all cows are the same colour. □

B. Submitted Questions

Question B.1. Prove that if $n \in \mathbb{N}$ then

$$\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}.$$

Question B.2. Let F_n be the Fibonacci sequence. Prove that

$$F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1.$$

C. Challenge Questions

Question C.1. Prove that if $x > -1$ and $n \in \mathbb{N}$ then $(1+x)^n \geq 1 + nx$.

Question C.2. Prove for all integers $n \in \mathbb{N}$ that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$