MTH307 - HOMEWORK 3

Solutions to the questions in Section B should be submitted by the start of class on 4/10/18.

A. Warm-up Questions

Question A.1. Prove the following by contradiction.

(i) There is no integer n which is both odd and even.
(ii) There are no integers m and n such that $21m + 30n = 1$.
(iii) $\sqrt[3]{2}$ is irrational.
(iv) If $\pi/2 \leq x \leq \pi$ then $\sin x - \cos x \geq 1$.

Question A.2. Prove the following with the contrapositive.

(i) Let a, b and c be integers. If a does not divide bc, then a does not divide b.
(ii) Let $x \in \mathbb{Z}$. If $x^3 - 1$ is even then x is odd.
(iii) Let $x \in \mathbb{R}$. If $x^5 + 3x^3 + x \geq x^4 + 6x^2 + 4$ then $x \geq 0$.
(iv) Let $x, y \in \mathbb{Z}$. If $x^2(y^2 - 2y)$ is odd then x and y are odd.

Question A.3. Prove or disprove the following.

(i) Let a and b be positive integers. Then if $a \mid b$ and $b \mid a$ then $a = b$.
(ii) If x and y are irrational then xy is irrational.
(iii) If $x \neq 0$ and x is rational and y is irrational then xy is irrational.
(iv) If $x, y \in \mathbb{Z}$ then $x^2 - 4y \neq 2$.
(v) Let $a, b \in \mathbb{R}$. Then if $a^3 + ab^2 \leq b^3 + a^2b$ then $a \leq b$.
(vi) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})[xy = xz]$.
(vii) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(\exists z \in \mathbb{R})[xy = xz]$.
(viii) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[xy = 1]$.
(ix) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})[xy = 1]$.
(x) $(\forall x \in \mathbb{Q})(\forall y \in \mathbb{Q})(\exists z \in \mathbb{Q})(x < z < y)$.

B. Submitted Questions

Question B.1. Prove or disprove the following.

(i) If $x \neq 0$ and x is rational and y is irrational then xy is irrational.
(ii) Let $x, y \in \mathbb{Z}$. Then if $x + y$ is even then x and y have the same parity.

C. Challenge Questions

Question C.1. Let $x \in \mathbb{R}$. Prove that $\sqrt{3} + x$ is irrational or $\sqrt{3} - x$ is irrational.

Question C.2. Let $x \in \mathbb{R}$ and suppose that for all $\varepsilon > 0$ we have $|x| < \varepsilon$. Prove that $x = 0$.

Question C.3. *A preview of analysis.* Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. We say that $f(x)$ *converges to* L as x converges to c if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$. In this case we write $\lim_{x \to c} f(x) = L$.

(i) Write out the negation of the above definition.
(ii) Let

$$f(x) = \begin{cases} 7 - 3x & \text{if } x \neq 1, \\ 2018 & \text{if } x = 1. \end{cases}$$

Prove that $\lim_{x \to 1} f(x) = 4$.

(iii) Let

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Prove that $\lim_{x \to 0} f(x) \neq 1$.