MTH307 - HOMEWORK 10

Here are some practice problems (mainly on cardinality) to help you prepare for the final.

Question 1. Suppose X, Y, Z are non-empty. Prove the following.

(i) $|X| = |X|$.
(ii) If $|X| = |Y|$ then $|Y| = |X|$.
(iii) If $|X| = |Y|$ and $|Y| = |Z|$ then $|X| = |Z|$.

Question 2. Let $|X| = n$. How many bijections are there on X?

Question 3. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ and $Y = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Prove that $|X| = |Y|$. Then use this to prove that $|[0,1)| = |[0,1]|$.

Question 4. Let X be an infinite set and $x \in X$. Prove that $X - \{x\}$ is infinite.

Question 5. Let X be an infinite set. Prove that X contains a countably infinite subset.

Question 6. Let A and B be finite sets. Prove the following.

(i) $A \cap B$ is finite.
(ii) $|A \cup B| = |A| + |B| - |A \cap B|$.
(iii) $|\{f \mid f\colon A \to B\}| = |B|^{|A|}$.

Question 7. Let X be countably infinite. Prove there exists $Y \subseteq X$ with $X \neq Y$ and $|X| = |Y|$.

Question 8. Prove that $\{n \in \mathbb{Z} \mid 7 \mid n\}$ is countably infinite.

Question 9. Suppose $|A| = |B|$. Prove that $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

Question 10. Let A and B be finite with $|A| = |B|$ and suppose $f\colon A \to B$. Show that f is injective if and only if it is surjective.

Question 11. Show that $\{X \in \mathcal{P}(\mathbb{N}) \mid X \text{ is finite}\}$ is countably infinite.