MTH307 - HOMEWORK 1

Solutions to the questions in Section B should be submitted by the start of class on 9/20/18.

A. Warm-up Questions

Question A.1. Show that the following pairs of statements are logically equivalent.

(i) $P \Rightarrow Q$ and $(\neg P) \vee Q$.
(ii) $P \vee Q$ and $Q \vee P$.
(iii) P and $(\neg P) \Rightarrow (Q \wedge (\neg Q))$
(iv) $(P \wedge Q) \vee R$ and $(P \vee R) \wedge (Q \vee R)$
(v) $P \wedge (Q \wedge R)$ and $(P \wedge Q) \wedge R$
(vi) $P \Rightarrow Q$ and $(\neg Q) \Rightarrow (\neg P)$

Question A.2. Prove DeMorgan's Laws. That is, show the following are logically equivalent.

(i) $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$.
(ii) $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$.

Question A.3. A *tautology* is a statement that is true no matter the truth values of the statement letters that occur in it. A *contradiction* is a statement that is false no matter the truth values of the statement letters that occur in it. Decide if the following are tautologies, contradictions or neither.

(i) $P \vee (\neg P)$.
(ii) $P \Rightarrow P$.
(iii) $P \wedge (\neg P)$.
(iv) $P \Leftrightarrow (\neg P)$.
(v) $P \Rightarrow (Q \Rightarrow P)$.
(vi) $(P \wedge (\neg Q)) \vee ((\neg P) \wedge Q))$.

Question A.4. Find negations the following sentences and write them out in clear, grammatical English.

(i) For all $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n \geq x$.
(ii) For all prime numbers p, there exists a prime number q with $q > p$.
(iii) There exists $b \in \mathbb{Z}$ such that for all $a \in \mathbb{Z}$, $ab = a$.
(iv) If $x^2 > 1$ then $x > 1$ or $x < -1$.
(v) If f is differentiable at c and f attains a local maximum at c, then $f'(c) = 0$.

B. Submitted Questions

Question B.1. Decide if $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge R$ are logically equivalent and justify your answer.

Question B.2. Negate the following sentence and write it out in clear, grammatical English. (The sentence is referring to a fixed sequence (a_n) of real numbers).

For all $\varepsilon > 0$ there exists a natural number K such that $|a_n| < \varepsilon$ whenever $n \geq K$.

C. Challenge Questions

Question C.1. Decide if the following are logically equivalent.

(i) $(\neg P) \Leftrightarrow Q$ and $(P \Rightarrow \neg Q) \wedge (\neg Q \Rightarrow P)$.
(ii) $P \Rightarrow (Q \Rightarrow R)$ and $(P \Rightarrow Q) \Rightarrow R$.
(iii) $(P \wedge (\neg Q)) \vee (Q \wedge (\neg P))$ and $(P \vee Q) \wedge (\neg(P \wedge Q))$.
(iv) $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ and $(P \wedge Q) \Rightarrow R$.

Question C.2. Define the logical connective $*$ by the formula $P * Q \equiv (\neg P) \wedge (\neg Q)$.

(i) Show that $\neg P \equiv P * P$
(ii) Show that $P \wedge Q \equiv (P * P) * (Q * Q)$
(iii) Show that $P \vee Q \equiv (P * Q) * (P * Q)$
(iv) Show that $P \Rightarrow Q \equiv ((P*P)*Q)*((P*P)*Q)$
(v) Is it true that $(P*Q)*R \equiv P*(Q*R)$?

Conclude that we can write all logical statements just using the operator $*$. Also conclude that this may be more trouble than its worth.