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Fractal Matings: When Two Polynomials Love Each Other Very Much

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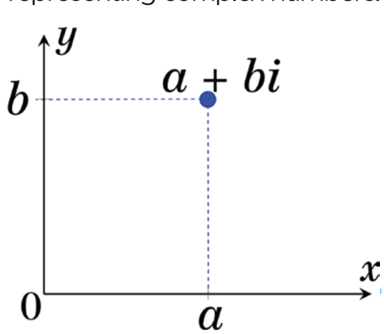
You have likely seen pictures of *fractals*—intricate objects with highly detailed geometry. Some well-known fractals are Julia sets. These fractals arise from the *iteration*, that is, *repeated application*, of complex polynomials on the set of complex numbers, which can be represented by the *Argand plane* as pictured in figure 1. Each point (a,b) in the plane corresponds to the complex number $a+ib$ where $i^2=-1$ is the imaginary unit.

For example, consider the complex-valued function $g(z)=z^2-1$. By iterating g on the real input $z=2$, we get $g(2)=3$, $g(g(2))=g(3)=8$, $g(g(g(2)))=g(8)=63$ and so on. Because the resulting outputs get larger and larger, we say that the input $z=2$ *escapes to infinity*. However, if we iterate g on the input $z=0$, we have $g(g(0))=0$, so that 0 returns to itself after two applications of g. We say that 0 is *periodic* of period 2: repeated application of g starting with 0 only bounces between 0 and $g(0)=-1$. Because 0 does not generate an unbounded collection of outputs, the input $z=0$ does not escape to infinity, and we say that 0 belongs to *the filled Julia set of g* (consequently, the input $z=-1$ also belongs to the filled Julia set of g).

Figure 1. The Argand plane representing complex numbers.



In general, given a complex-valued function f, if an input $z=a+ib$ does not escape to infinity upon repeated application of f, we say $a+ib$ belongs to the filled Julia set of f. We can have a computer color the corresponding points in the Argand plane to give a visual representation of the filled Julia set. The *Julia set of f* is then the boundary of the filled Julia set of f. Figure 2 shows the filled Julia set of the function $g(z)=z^2-1$, which is referred to as the *Basilica*.

The Rabbit

One particular filled Julia set, the (Douady) *Rabbit*, will serve as the protagonist of our discussion about matings. This filled Julia set is named after Adrien Douady—one of the fathers of the modern theory of complex dynamical systems and one of the first people to use computer graphics to study Julia sets. The term *mating* used in this article is due to Douady and his collaborator John H. Hubbard.

The Rabbit arises as the filled Julia set of the polynomial $r(z)=z^2+c$, where $c\approx-0.12256117+0.74486177i$, pictured in figure 3. The central red region (containing 0) represents the head, and the two smaller red regions coming off the top of this region look like ears. This set constitutes one of the standard "first examples" used in studying the dynamics of polynomials. In particular, this filled Julia set is perhaps the simplest example for which the associated value of c is not a real number: the behavior of the function is relatively simple but is more complicated than the examples we would see if we only considered real values of c. The distinctive imagery associated with the Rabbit also makes it easier to spot (with its head and two ears)

Figure 2. The filled Julia set (top) for $g(z) = z^2 - 1$, which is known as the Basilica due to its resemblance to the Basilica di San Marco in Venice (bottom). The arrows indicate the period 2 point.

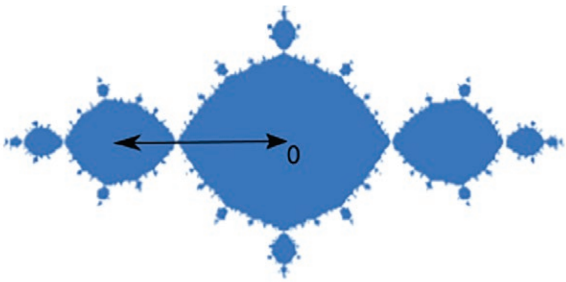
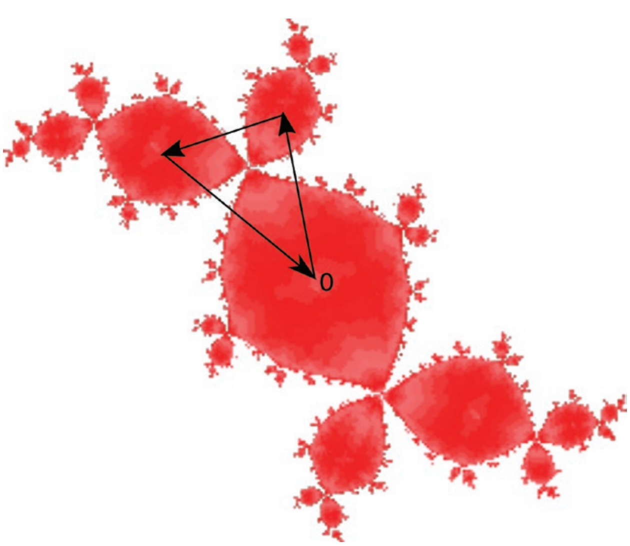


Photo from Zairon, used under CC BY-SA 4.0; *commons.wikimedia.org/wiki/File:Venezia_Basilica_di_San_Marco_Fassade_5.jpg*

Figure 3. The Douady Rabbit filled Julia set. The period 3 orbit is represented by arrows.



when it appears in connection with other Julia sets, a property that we will use to our advantage.

In the study of dynamical systems, it is often helpful to analyze the behavior of a function f at its critical points, that is, the set of points where $f'(z) = 0$.

Because $r'(z) = 2z$, the Rabbit's only critical point is the input $z = 0$. One interesting feature of the Rabbit is that the critical point $z = 0$ has period 3: three applications of the map to $z = 0$ results in 0, that is, $r(r(r(0))) = 0$. Everything inside the red region eventually gets pulled towards the cycle made up of 0, $r(0)$, and $r(r(0))$, and every other point (in the white region in the plane outside the Rabbit), escapes to infinity!

Rational Maps

The Basilica and the Rabbit both come from functions of the form $f(z) = z^2 + c$, but we can build the filled Julia set for more complicated functions too. For instance, figure 4 takes us on an expedition through various filled Julia sets of rational maps of the form $F(z) = (z^2 + u) / (z^2 + v)$ for appropriate choices of complex numbers u and v. These maps are part of the more general family of *rational maps* of the form $F(z) = p(z) / q(z)$ where both $p(z)$ and $q(z)$ are polynomials.

If you look closely at the filled Julia sets in figure 4, you may see a familiar shape. Notice that the "ears" of the rabbit appear in each of these Julia sets. Some of them are harder to see (can you spot them all?). These images honestly represent genuine pictures of Julia sets of rational functions, so why is it that they all contain the Rabbit?

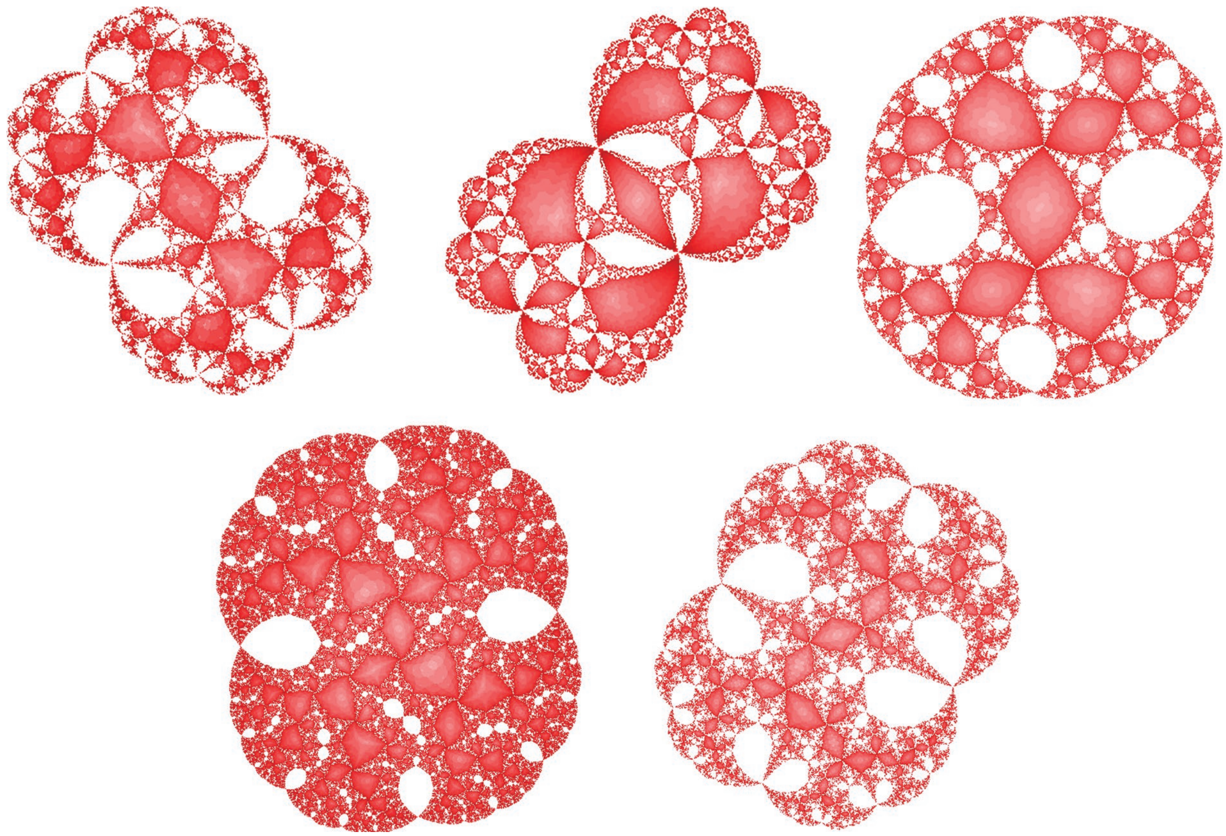
It turns out that we can explain this phenomenon: all of the rational maps were formed by "mating" the Rabbit with another polynomial Julia set (see the corresponding mates on the supplementary page *maa.org/mathhorizons/supplemental.htm).* But there is more. Not only do the rational maps *look* like the Rabbit, they also *behave* like the Rabbit. The way the maps behave under iteration is very similar to that of the Rabbit polynomial. After all, mating does not just pass down physical characteristics such as one's appearance. We also see more intrinsic traits passed down onto offspring.

Matings

To perform a mating of two polynomials, we essentially glue the two corresponding filled Julia sets together along their boundary. To do this, we need to find a way of identifying points on the Julia sets of the two polynomials so that we know which points to glue. The first step requires us to assign to each point in the Julia set at least one point t on the unit circle. The rigorous mathematical way of doing this requires clever techniques that we won't describe here. However, we will give a nice physical interpretation of how this is done.

Suppose we find a skilled metalworker who can build a model of the Julia set for us. After paying them handsomely and thanking them for a job well done, we take the Julia set, place it on the Argand plane centered at the origin, and give the Julia set a

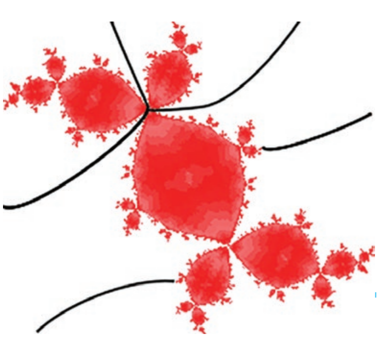
Figure 4. Some filled Julia sets for rational maps. What do they all have in common?



positive electrical charge. For each value t between 0 and 1, trace a straight line from the origin creating an angle of $2\pi t$ radians from the real axis. Once you have extended the line far enough beyond the Julia set, place an electron at the end of the line in the plane. Being negatively charged, the electron will be attracted towards the positively charged Julia set, and the electron will trace out a path in the plane called the *external ray of angle* $2\pi t$. The point on the Julia set where the electron lands is the point to which we assign the angle $2\pi t$.

Figure 5 shows a schematic diagram of this process for the Rabbit with a few typical external rays landing on the Julia set. For the Rabbit, we call the point where the electron lands $R(t)$. Notice that some points of the Julia set have more than one external ray landing on them! That is, they can be approached by electrons from more than one direction, and so we may have $R(t) = R(s)$ even though $s \neq t$.

Figure 5. The Rabbit Julia set with a few external rays landing on it.



Equipped with this mating process, we are ready to see how the mischievous Rabbit has combined its DNA with other polynomials to create the filled Julia sets in figure 4.

An Example Mating

Let's mate our hero, the Rabbit, with the Basilica, pictured in figure 1. As we did with the Rabbit, we can assign to each point of the Julia set of the Basilica an associated angle t, which we'll denote by $B(t)$. Then, to glue the two sets together, we stick the Basilica onto the outside of the Julia set of the Rabbit: turn the Basilica "inside-out" and glue the point $R(t)$ to the point $B(-t)$. Here, we use $B(-t)$ instead of $B(t)$ because we want to turn the Basilica "inside-out," which reverses the direction of the angles. Including the negative sign on the t allows everything to match up correctly and reverses the effect of "turning inside-out." Figure 6 shows the Basilica turned inside-out.

To think about this process, imagine, as in figure 7, that we place the rabbit inside the inverted basilica, and then pull the two Julia sets together so that the intermediate white space in between disappears. The resulting set certainly looks like it could be the Julia set of a rational map. In fact, this set appears in figure 4 and is the filled Julia set for the rational map $F(z) = (z^2 + \frac{\sqrt{3}+i}{2}) / (z^2 - 1)$.

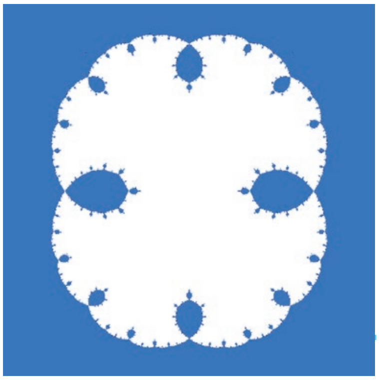


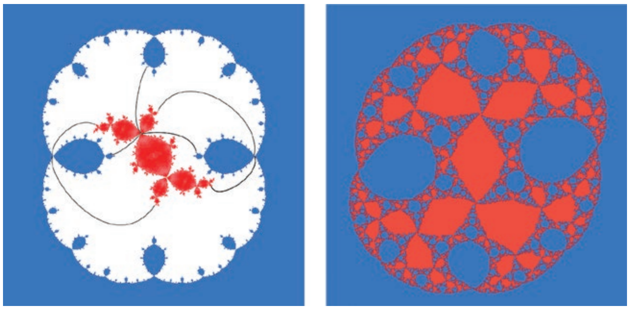
Figure 6. The Julia set of the Basilica turned inside-out.

Unfortunately, the rational map for the mating is *not* the quotient of the two polynomials involved—if only life were that easy! For instance, the last example from figure 4 confirms that the rational map cannot be such a quotient: that Julia set results from the rational map that is the mating of the rabbit with a copy of itself. In this case, the quotient of the two polynomials would be $F(z) = 1$, which is surely not the case, as the filled Julia set would be the entire plane.

While the mating doesn't preserve the coefficients of the polynomials, it does preserve more than just the physical appearances of the two Julia sets. It turns out that the dynamical behavior has been carried across as well!

Recall that the Rabbit has a period 3 critical point. One of the critical points of the rational map for the mating returns to itself after three iterates. Thus, the Rabbit has passed on not just its physical traits under the mating, but also its dynamical traits. The same is true for the Basilica; the other critical point of the rational map returns to itself after two iterates, just like the critical point of the Basilica. The Rabbit and the Basilica have combined to create a beautiful, bouncing baby Julia set. Now we should leave them alone and let the proud parents care for their new bundle of joy.

Figure 7. The mating of the Basilica and the Rabbit. The black curves show some of the points that are glued together under the mating.



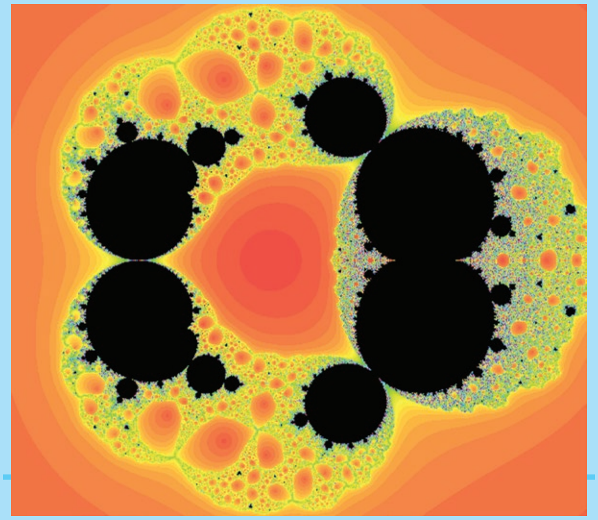
MATINGS IN RESEARCH

Complex dynamics researchers study matings because it allows them to describe the (complicated) dynamics of rational maps in terms of the (simpler) dynamics of polynomials. The dynamics of polynomials are fairly well (but by no means completely!) understood. On the other hand, the dynamics of rational maps are still a source of many interesting questions in complex dynamics. It is actually possible to draw a picture of the "space" of rational maps that have a period 3 critical point. When we do this in figure 8, we see a familiar shape (look at the top middle and bottom for some ears).

That rascally rabbit! This article has only scratched the surface of questions about polynomial matings. For example, when is it possible to combine two polynomials in a mating to make a rational map? Which rational maps are matings, and which ones are not? Are there other ways of combining polynomials to make rational maps (this is a problem currently being studied)?

As an added bonus, studying matings means researchers get to look at lots of pretty pictures of Julia sets of rational maps.

Figure 8. A representation of the space of rational maps that have a period 3 critical point.



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