A CLASSIFICATION OF BICRITICAL RATIONAL MAPS WITH A PAIR OF PERIOD TWO SUPERATTRACTING CYCLES.

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Abstract. We give a Thurston classification of those bicritical rational maps which have two period two superattracting cycles. We also show that all such maps are constructed by the mating of two unicritical degree d polynomials.

1. Introduction

In this note we study bicritical rational maps whose critical points lie in distinct period two cycles. These maps are completely classified by a natural combinatorial invariant. A common strategy used in the classification of rational maps is to invoke Thurston's Theorem [DH93]. This will not be necessary in this note, due to the availability of a purely algebraic classification of the maps in question. However, we will still tackle the question of Thurston equivalence at the end of the paper, since it motivates a discussion of the combinatorial description of these maps, which is a characteristic that will be much more useful than the algebraic classification in general. We refer the reader to [Mil06] for background material on the dynamics of rational maps.

1.1. **Thurston's Criterion.** Let $F\colon \Sigma \to \Sigma$ be an orientation-preserving branched self-covering of a topological 2-sphere. We denote by Γ_F the critical set of F and define

$$P_F = \bigcup_{n>0} F^{\circ n}(\Gamma_F)$$

to be the postcritical set of F. We say that F is postcritically finite if $|P_F| < \infty$.

Definition 1.1. Let $F\colon \Sigma \to \Sigma$ and $\widehat{F}\colon \widehat{\Sigma} \to \widehat{\Sigma}$ be postcritically finite orientation-preserving branched self-coverings of topological 2-spheres. A *combinatorial equivalence* (or Thurston equivalence) is given by a pair of orientation-preserving homeomorphisms (Φ, Ψ) from Σ to $\widehat{\Sigma}$ such that

- $\Phi|_{P_F} = \Psi|_{P_F}$
- The following diagram commutes:

$$\begin{array}{ccc} (\Sigma, P_F) & \xrightarrow{\ \Psi\ } & (\widehat{\Sigma}, P_{\widehat{F}}) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{F}} \\ (\Sigma, P_F) & \xrightarrow[\ \Phi\ ]{} & (\widehat{\Sigma}, P_{\widehat{F}}) \end{array}$$

- Φ and Ψ are isotopic via a family of homeomorphisms $t \mapsto \Phi_t$ which is constant on P_F.

We say F and $\widehat{F}$ are equivalent if there exists an equivalence as above. Given a branched covering F, we want to know when it is equivalent to a rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Let $\Gamma = \{\gamma_i : i = 1, \dots, n\}$ be a multicurve; that is, each $\gamma_i \in \Gamma$

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is simple, closed, non-peripheral, disjoint and non-homotopic to all the other γ_j relative to P_F. Γ is *F-stable* if for all $\gamma_i \in \Gamma$, all the non-peripheral components of $F^{-1}(\gamma_i)$ are homotopic rel $\Sigma \setminus P_F$ to elements of Γ. In this case, we define $F_\Gamma = (f_{ij})_{n\times n}$ to be the non-negative matrix defined as follows. Let $\gamma_{i,j,\alpha}$ be the components of $F^{-1}(\gamma_j)$ which are homotopic to γ_i in $\Sigma \setminus P_F$. Now define

$$F_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{\deg F|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j} \gamma_i$$

where deg denotes the degree of the map. By standard results on non-negative matrices (see [Gan59] or [Sen81]), this matrix (f_{ij}) will have a leading non-negative eigenvalue λ. We write $\lambda(\Gamma)$ for the leading eigenvalue associated to the multicurve Γ.

The significance of these definition lies in the following rigidity theorem of Thurston.

Theorem 1.2 (Thurston)**.**

(1) *A postcritically finite branched covering $F\colon \Sigma \to \Sigma$ of degree $d \geq 2$ with hyperbolic orbifold is equivalent to a rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ if and only if there are no F-stable multicurves with $\lambda(\Gamma) \geq 1$*[1].

(2) *Any Thurston equivalence of rational maps F and $\widehat{F}$ with hyperbolic orbifolds is represented by a Möbius conjugacy.*

Remark 1.3.

- The condition that the maps have hyperbolic orbifolds is purely combinatorial and is readily ascertained by inspection of the postcritical set. The condition will be satisfied by all the maps in this paper. For further details see [DH93].
- For bicritical maps, any such conjugacy (indeed, any equivalence) either fixes the two critical points or interchanges them. In our setting, the postcritical set has (at least) four elements, and so any Möbius conjugacy fixing the two critical points must be the identity.

Before moving on, we need to introduce a special kind of multicurve, called a (good) Levy cycle.

Definition 1.4. A multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a Levy cycle if for each $i = 1, \ldots, n$, the curve γ_{i-1} (or γ_n if $i = 1$) is homotopic to some component γ_i' of $F^{-1}(\gamma_i)$ (rel P_F) and the map $F\colon \gamma_i' \to \gamma_i$ is a homeomorphism. A Levy cycle is a good Levy cycle if the connected components of $\Sigma \setminus \bigcup_{i=1}^n \gamma_i$ are $D_1, \ldots, D_m, C$, with the D_j all being disks. When $n = 1$, we have $C = \varnothing$ and $F|_{\gamma_1'}\colon \gamma_1' \to \gamma_1$ reverses the orientation. For $n > 1$, there exists a component C' of $F^{-1}(C)$ which is isotopic to C and such that $F|_{C'}\colon C' \to C$ is a homeomorphism.

It is clear that a good Levy cycle L is a Thurston obstruction, since $\lambda(L) = 1$.

1.2. **Matings.** In this section, we review the basic terminology of matings, following [Mil04, ST00, Tan92]. Let f and g be monic degree d polynomials. In this paper, f and g will be unicritical (that is, taking the form $z \mapsto z^d + c$ for some c) but this is not needed in general. We define

$$\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2\pi i t} : t \in \mathbb{R}/\mathbb{Z}\},$$

the complex plane compactified by the circle at infinity. We then continuously extend the two polynomials to the circle at infinity by defining

$$f(\infty \cdot e^{2\pi i t}) = \infty \cdot e^{2d\pi i t} \quad \text{and} \quad g(\infty \cdot e^{2\pi i t}) = \infty \cdot e^{2d\pi i t}.$$

[1]Such a multicurve is known as a *Thurston obstruction*

Label this extended dynamical plane of f (respectively g) by $\widetilde{\mathbb{C}}_f$ (respectively $\widetilde{\mathbb{C}}_g$). We create a topological sphere $\Sigma_{f,g}$ by gluing the two extended planes together along the circle at infinity:

$$\Sigma_{f,g} = \widetilde{\mathbb{C}}_f \uplus \widetilde{\mathbb{C}}_g / \sim$$

where $\sim$ is the relation which identifies the point $\infty \cdot e^{2\pi it} \in \widetilde{\mathbb{C}}_f$ with the point $\infty \cdot e^{-2\pi it} \in \widetilde{\mathbb{C}}_g$. The *formal mating* is then defined to be the branched covering $f \uplus g \colon \Sigma_{f,g} \to \Sigma_{f,g}$ such that

$$f \uplus g|_{\widetilde{\mathbb{C}}_f} = f \quad \text{and}$$
$$f \uplus g|_{\widetilde{\mathbb{C}}_g} = g.$$

The definition of the topological mating requires discussion of external rays. Suppose the (filled) Julia set $K(f)$ of the degree $d \geq 2$ monic polynomial $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is connected (this will always be the case in this article). By Böttcher's theorem, there is a conformal isomorphism

$$\phi = \phi_f \colon \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K(f)$$

which can be chosen so that it conjugates $z \mapsto z^d$ on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with the map f on $\widehat{\mathbb{C}} \setminus K(f)$. There is a unique such conjugacy which is tangent to the identity at infinity.

Definition 1.5. The external ray of angle t is

$$R_f(t) = \phi_f(r_t)$$

where $r_t = \{r \exp(2\pi it) : r > 1\} \subset \mathbb{C} \setminus \mathbb{D}$ is the corresponding radial line.

Recall that $K(f)$ (equivalently $J(f)$) is locally connected if and only if ϕ extends continuously to the unit circle, thereby inducing a semi-conjugacy $\gamma_f \colon \mathbb{R}/\mathbb{Z} \to J(f)$. The point $\gamma_f(t) \in K(f)$ is called the *landing point* of the external ray $R_f(t)$. The points $k/(d-1)$ are fixed under the map $t \mapsto dt$ on $\mathbb{R}/\mathbb{Z}$ with images $\beta_k = \gamma_f(k/(d-1))$ fixed under f. We call these the β-fixed points of f_c. In the case that these β_k are distinct, there exists at most one other fixed point, which we will call the α-fixed point of f_c.

Lemma 1.6. *Suppose f is a unicritical polynomial. Then the β-fixed points of f are all distinct.*

Proof. We consider f as a map on $\widetilde{\mathbb{C}}$ and suppose that $\beta_j = \beta_k$ for some $j \neq k$. The external rays of angle $\hat{\beta}_j$ and $\hat{\beta}_k$ and their common landing point divide the disk into two sectors. Denote by U the one which does not contain the critical point. The smallest possible angular width of the sector U is evidently $1/(d-1) > 1/d$ whence the image of the sector under f must be the whole of the disk $\widetilde{\mathbb{C}}$. However since the critical point is not contained in U, the critical value cannot be in its image $f(U)$. Hence all the β_i must be distinct. □

We now define the topological mating of monic degree d polynomials f and g with locally connected Julia sets. We first define the ray-equivalence relation $\approx$ on $\Sigma_{f,g}$. We denote by $\sim_f$ the smallest equivalence relation on $\widetilde{\mathbb{C}}_f$ such that $x \sim_f y$ if and only if $x, y \in \overline{R}_f(t)$ for some t; we denote by $\sim_g$ the corresponding equivalence relation on $\widetilde{\mathbb{C}}_g$. Then the equivalence relation $\approx$ is the smallest equivalence relation on $\Sigma_{f,g}$ generated by $\sim_f$ on $\widetilde{\mathbb{C}}_f$ and $\sim_g$ on $\widetilde{\mathbb{C}}_g$. We will denote the equivalence class of x under $\approx$ by $[x]$ and by $[x]_J$ the intersection of the Julia sets of the two polynomials with $[x]$. We now observe that there is an evident surjection $K(f) \uplus K(g) \to \Sigma_{f,g}/\approx$ formed by the composition

$$K(f) \uplus K(g) \hookrightarrow \widetilde{\mathbb{C}}_f \uplus \widetilde{\mathbb{C}}_g \to \Sigma_{f,g} \to \Sigma_{f,g}/\approx.$$

Moreover, since $x \approx y$ implies $f \uplus g(x) \approx f \uplus g(y)$, there exists a unique map $f \perp\!\!\!\perp g$ such that the following diagram commutes:

$$\begin{array}{ccc} K(f) \uplus K(g) & \xrightarrow{f \uplus g} & K(f) \uplus K(g) \\ \big\downarrow & & \big\downarrow \\ \Sigma_{f,g}/\approx & \xrightarrow[f \perp\!\!\!\perp g]{} & \Sigma_{f,g}/\approx . \end{array}$$

We refer to this map $f \perp\!\!\!\perp g$ as the topological mating of f and g. When circumstances are favorable, the quotient space $\Sigma_{f,g}/\approx$ is an oriented topological 2-sphere.

We say that a rational map F is the geometric mating of f and g if F is topologically conjugate via an orientation-preserving homeomorphism $\varphi\colon \Sigma_{f,g}/\approx\ \to \widehat{\mathbb{C}}$ which is holomorphic on the interior of $K_f \uplus K_g$. In this situation, we will write $F \cong f \perp\!\!\!\perp g$. It is important to realize that the same rational map may arise as a mating in several different ways; as will in fact be the case in this paper. This state of affairs is referred to as a *shared mating*. The following key result on the matings of unicritical polynomials will be of use in this article.

Theorem 1.7 (Tan Lei, [Tan87])**.** *Let f_1 and f_2 be unicritical postcritically finite polynomials with α-fixed points α_1 and α_2 respectively. Then the following are equivalent mating $f_1 \perp\!\!\!\perp f_2$ is equivalent to a rational map if and only if $[\alpha_1] \neq [\alpha_2]$.*

1.3. **Structure.** Our classification will go as follows. First, we will consider all matings $f \perp\!\!\!\perp g$ where f and g both are monic unicritical polynomials with a critical orbit of period two. We then show that such matings, when unobstructed, yield rational maps whose Julia sets admit a simple combinatorial description. Finally, by using a simple counting argument, we show that these rational maps account for all the rational maps that have a pair of period two superattracting cycles. Our main result is the following.

Theorem A. *A degree d bicritical rational map with labeled critical points and with two period two superattracting cycles may be realized as a mating in precisely $d-1$ ways. Furthermore, the rational map is completely defined (up to Möbius conjugacy) by one piece of combinatorial data.*

Despite the relative simplicity of the result, perhaps the most striking observation is that the rational maps with a pair of period two superattracting cycles have a common feature to their Julia sets, as will be seen in Section 3. It was this observation that lead to the conjecture of Theorem A.

Remark 1.8. The rational maps in question actually have *clusters*, as studied in the second author's thesis [Sha11]. More general results on maps with cluster cycles can be found in [Sha12a, Sha12b].

At the end of this paper, we will show that the result could equally well be obtained by demonstrating the Thurston equivalence of the maps with the same combinatorial datum, which would allow us to avoid an explicit algebraic description as in Section 2. This second approach more readily adapts to the general setting of higher degree maps with cluster cycles, hence it is included here. Indeed, it should be noted that in general, clustering is not an algebraic condition; however, in the setting of this paper, it is a fortunate and perhaps surprising fact that the algebraic classification suffices, hence there is no requirement to check the Thurston equivalence of these maps.

2. Algebraic Classification

We remark that it is simple to classify the degree d rational maps with two superattracting period two cycles in algebraic terms. Setting the two critical points to be 0 and ∞ and defining $F(\infty) = 1$, we obtain the following normal form:

$$F_a(z) = \frac{z^d - a}{z^d - 1}. \tag{1}$$

Note that $F_a(0) = a$, so the critical values of F_a are 1 and a. For there to be a second period two cycle, we require

$$0 = F_a(a) = \frac{a^d - a}{a^d - 1} = \frac{a(a^{d-1} - 1)}{a^d - 1}.$$

It follows that in order for the map to have a second period two superattracting cycle, we need $a \neq 1$ and for a to be a $(d-1)$th root of unity. Hence there are precisely $d - 2$ parameters a for which F_a has a pair of period two superattracting cycles.

3. Matings Classification

3.1. **Structure of the polynomials.** Let $\mathcal{M}_d$ be the degree d multibrot set. That is, for $\mathrm{P}_c \colon z \mapsto z^d + c$,

$$\mathcal{M}_d = \{c \in \mathbb{C} : J(\mathrm{P}_c) \text{ is connected}\} = \{c \in \mathbb{C} : (\mathrm{P}_c^{\circ n}(0))_{n=0}^{\infty} \text{ is bounded}\}.$$

For notation, we will write

$$\hat{\beta}_j = \frac{j}{d-1} \quad \text{and} \quad \theta_j^r = \hat{\beta}_j + \frac{r}{d^2 - 1},$$

with subscripts taken modulo $d - 1$ and $r \in \{1, \ldots, d\}$. Note that the angles $\hat{\beta}_j$ are fixed under the map $z \mapsto dz$ on S^1; indeed, these angles are those of the rays which land on the β-fixed points of the degree d polynomials under consideration, so that $\beta_j = \gamma_f(\hat{\beta}_j)$. Moreover, the angles θ_j^r all have period 2. By results of [Sch99] (or by adapting the results of [Mil00] to the degree d case), there is a unique $c_j \in \widehat{\mathbb{C}}$ such that $f_j = \mathrm{P}_{c_j}$ has a period two superattracting cycle and such that the external rays of angles θ_{j-1}^1 and θ_{j-1}^d land on the α-fixed point α_j of f_j. Figure 1 shows the location of these maps in parameter space. Note that the α-fixed point of f_j is also the principal root point of both the critical point and critical value Fatou components. We prove some elementary facts about the f_j.

Lemma 3.1.
(1) *A point of $J(f_j)$ is biaccessible if and only if it is in the backward orbit of α_j.*
(2) *The only biaccessible periodic point of f_j is α_j.*

Proof. For the first claim, if z is biaccessible then it must eventually map onto the Hubbard tree of f_j. In this case, the Hubbard tree consists of two internal rays meeting at the α-fixed point, whence the only point of the Hubbard tree that belongs to $J(f_j)$ is α_j, and so there exists some ℓ such that $f^{\circ \ell}(z) = \alpha_j$. Conversely, since α_j and the critical point belong to distinct grand orbits and since α_j is biaccessible, every point on the grand orbit of α_j is biaccessible. The second claim follows immediately from the first. □

We now discuss which periodic rays land on the periodic components of $J(f_j)$.

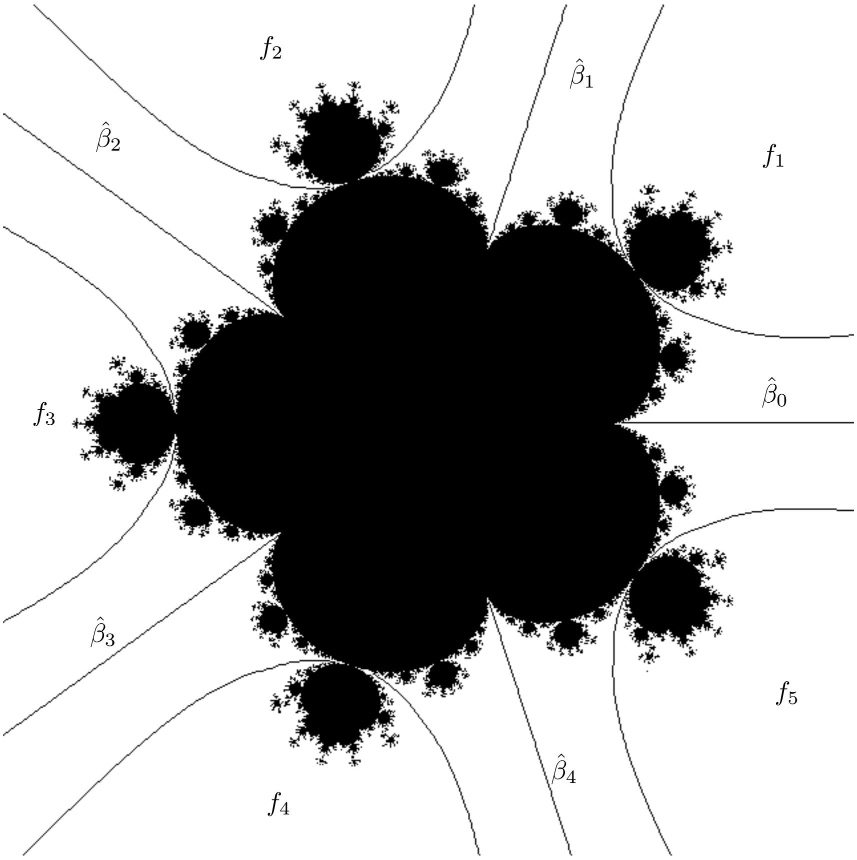

FIGURE 1. The maps f_j in the parameter space $\mathcal{M}_6$. The $\hat{\beta}_j$ are labeling the rays of angle $\hat{\beta}_j$ and the f_i labels are contained in the wake of the period two hyperbolic component which contains f_i.

Lemma 3.2.

(1) *The external rays of angles*

$$\theta^2_{j-1}, \theta^3_{j-1}, \ldots, \theta^r_{j-1}, \ldots, \theta^{d-1}_{j-1} \tag{2}$$

land on the critical value component of f_j.

(2) *The external rays of angles*

$$\theta^{d-1}_j, \theta^{d-2}_{j+1}, \ldots, \theta^{d-(r-1)}_{j+r-2}, \ldots, \theta^2_{j-1+(d-2)} = \theta^2_{j-2} \tag{3}$$

land on the critical point component of f_j.

Proof.

(1) The map $f_j^{\circ 2}$ is a degree d map from the critical value component to itself, and so there must exist precisely $d-1$ fixed points of $f_j^{\circ 2}$ on the boundary of this component. One of these is the α-fixed point and the others must be of period two, since the only fixed points of f_j are the α_j and the β_ℓ. These points must be the landing points of external rays whose period is divisible by two, and since a fixed ray cannot land on a period 2 point, the periods of the rays landing there is also exactly two. Furthermore, the angles of these rays must lie in the arc $(\theta^1_{j-1}, \theta^d_{j-1}) \subset S^1$, and the only period two

angles in this arc are

$$\theta_{j-1}^2, \theta_{j-1}^3, \ldots, \theta_{j-1}^r, \ldots, \theta_{j-1}^{d-1}$$

as required.

(2) The images under the map $z \mapsto dz$ of the angles in (2) are precisely those angles in (3), since

$$\begin{aligned} d(\theta_{j-1}^r) &= d\hat{\beta}_{j-1} + \frac{dr}{d^2-1} \\ &= \hat{\beta}_{j-1} + \frac{dr}{d^2-1} \\ &= \hat{\beta}_{j-1} + \frac{r-1}{d-1} + \frac{d-(r-1)}{d^2-1} \\ &= \hat{\beta}_{j+r-2} + \frac{d-(r-1)}{d^2-1} \\ &= \theta_{j+r-2}^{d-(r-1)}. \end{aligned}$$

Since the rays of the angles given in (2) land on the critical value component, the rays in (3) must land on the critical point component.

□

Following custom, we refer to the fixed points of the first return map to a periodic Fatou component as the *root* points of the component. If the root point is the landing point of more than one external ray, we will call it the *principal* root point. The other root points will be called *co-roots*. Here we see that α_j is the principal root point of both the critical value and critical point components, and that the landing points of the rays of angles given in (2) and (3) will land at the co-roots of the respective components. Roots of preperiodic components are accordingly defined as the preimages of roots of periodic components. See Figure 2 for an example of the rays landing on the Julia set in the case $d = 6$ and $j = 2$.

Lemma 3.3. *Let U be the critical point component of f_j and let U' be a component of $f^{-1}(U)$ that is not the critical value component. Then the external rays landing at the principal root point of U' separate the critical point 0 from precisely one of the fixed points β_r.*

Proof. Let $A \subset \mathbb{C}$ be the region which is bounded by the external rays landing at the principal root point, namely α_j, of the critical value component of f_j and the level $2d$ equipotential curve, but which does not contain the critical value component itself. Let g be the local inverse of f_j which takes U to U'. Then g is a homeomorphism and $\overline{g(A)} \subset A$, since $g(A)$ is bounded by the rays landing at the preimage of α_j on the boundary of U' and the level 2 equipotential curve. Hence by the Schwarz Lemma, g has a unique fixed point in $g(A)$. This fixed point is not α_j, since clearly $\alpha_j \notin g(A)$ and it is not ∞ since $g(A)$ is bounded. □

Note that the above result is equivalent to saying that the rays landing at the principal root point of U' separate the critical point 0 from (precisely) one the rays of angle $\hat{\beta}_r$.

3.2. **Matings and ray classes.** In this section we want to consider the ray classes of matings which are not obstructed. *For the matings, we will write $f_{i_+} \perp\!\!\!\perp f_{i_-}$, where the indices $i_+, i_- \in \{1, \ldots, d-1\}$.* For ease of notation, we will write γ_+ (respectively γ_-) for the semi-conjugacy $\gamma_{f_{i_+}}$ (respectively $\gamma_{f_{i_-}}$).

Lemma 3.4. *Suppose the mating $f_{i_+} \perp\!\!\!\perp f_k$ is not obstructed and $z \in J(f_{i_+}) \cup J(f_k)$. Then the ray class $[z]_J$ has one of the following forms.*

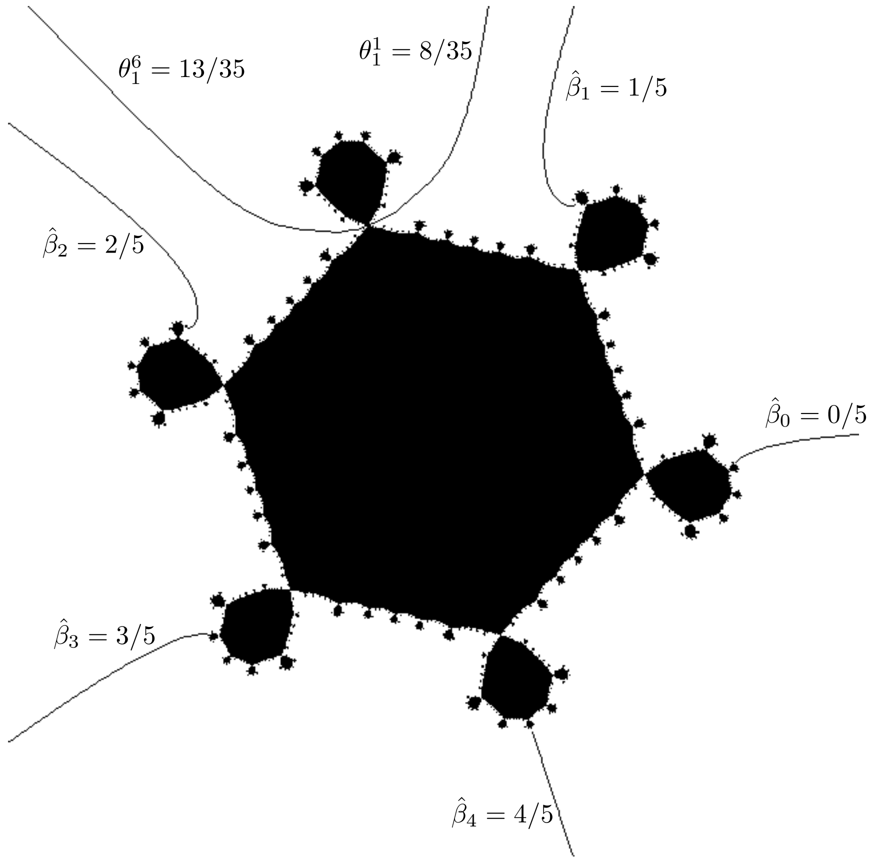

FIGURE 2. The Julia set of the map f_2 in degree 6 with some important dynamical rays labeled.

- *If z is in the backward orbit of α_{i_+} then $[z]_J = \{z, \gamma_k(-t_1), \gamma_k(-t_2)\}$ where t_1 and t_2 are the angles of the external rays landing at z.*
- *If z is in the backward orbit of α_k then $[z]_J = \{z, \gamma_j(-\tau_1), \gamma_j(-\tau_2)\}$ where τ_1 and τ_2 are the angles of the external rays landing at z.*
- *If z is not in the backward orbit of α_j or α_k then $[z]_J = \{\gamma_j(t), \gamma_k(-t)\}$ for some t.*

Proof. Note that $[\alpha_j] \neq [\alpha_k]$ by Theorem 1.7. We claim that if $z = \alpha_j$ then the ray class $[z]_J$ contains three points. Indeed, all points in $[\alpha_j]_J$ must be periodic of period dividing two and by Lemma 3.1 the only biaccessible periodic points of $J(f_j) \cup J(f_k)$ are α_j and α_k, whence the points $\gamma_k(-t_1)$ and $\gamma_k(-t_2)$ are not biaccessible. Thus $[\alpha_j]_J = \{\alpha_j, \gamma_k(-t_1), \gamma_k(-t_2)\}$ where t_1 and t_2 are the angles of the external rays which land at α_j. The case where $z = \alpha_k$ is analogous. Moreover, if z is in the backward orbit of α_j or α_k then the ray class $[z]_J$ also contains three points since neither α_J or α_k belong to the grand orbit of a critical point. If z is not in the backward orbit of α_j or α_k, then by Lemma 3.1, none of the points of $[z]_J$ will be biaccessible and so $[z]_J = \{\gamma_j(t), \gamma_k(-t)\}$ for some t. □

Lemma 3.5. *The mating $f_j \perp\!\!\!\perp f_k$ is obstructed if and only if $k = d - j$.*

Proof. If $k = d - j$ then clearly $[\alpha_j] = [\alpha_k]$. Conversely, by Lemma 3.4,

$$[\alpha_j]_J = \{\alpha_j, \gamma_k(-\theta_{j-1}^1), \gamma_k(-\theta_{j-1}^d)\}.$$

Now $\alpha_k \notin \{\gamma_k(-\theta^1_{j-1}), \gamma_k(-\theta^d_{j-1})\}$ since $k \neq d-j$. In particular $[\alpha_j] \neq [\alpha_k]$ and the proof of the lemma follows from Tan Lei's theorem (Theorem 1.7). $\square$

We can say more: the only periodic ray classes which contains more than two points of $J(f_j) \cup J(f_k)$ are $[\alpha_j]$ and $[\alpha_k]$.

Proposition 3.6. *Suppose $j+k \neq d$. Then the rational map $F \cong f_j \perp\!\!\!\perp f_k$ has the following properties.*

- *The closures of the two critical value components are disjoint.*
- *The closures of the two critical point components are disjoint.*

Proof. Note that the angles of the rays landing on the critical value component of f_j lie in the arc

$$(\theta^1_{j-1}, \theta^d_{j-1})$$

and the angles of the rays landing on the critical value component of f_k lie in the arc

$$(\theta^1_{k-1}, \theta^d_{k-1}).$$

Since $j+k \neq d$ and $-\theta^r_{k-1} = \theta^{d-r+1}_{d-k-1}$, it follows that

$$(\theta^1_{j-1}, \theta^d_{j-1}) \cap (-\theta^d_{k-1}, -\theta^1_{k-1}) = \varnothing. \tag{4}$$

By Lemma 3.4, all ray classes $[z]_J$ contain at most 3 points, and the result follows since any ray class corresponding to a meeting point between the two critical value components would contain a ray with angle in the intersection in (4), which is empty. $\square$

Lemma 3.7. *Let $F \cong f_j \perp\!\!\!\perp f_k$. Then the following conditions hold.*

- *The only ray class which meets the boundary of the critical point component of f_j and the critical value component of f_j is $[\alpha_j]$.*
- *The only ray class which meets the boundary of the critical point component of f_k and the critical value component of f_k is $[\alpha_k]$.*

Proof. We will prove the first claim; the proof of the second claim is entirely analogous. Since the critical point component and the critical value component of f_j have closures intersecting at α_j, the corresponding Fatou components of F, C_j (which contains the critical point) and V_j (which contains the critical value) meet at $[\alpha_j] \in J(F)$. We claim that this is the only meeting point. Indeed, any such point arises from a ray class which contains one boundary point from C_j and one boundary point from V_j. If the ray class is not $[\alpha_j]$, then it contains a biaccessible point of $J(f_k)$ and so, by Lemma 3.4, eventually maps onto $[\alpha_k]$. It follows that $[\alpha_k]$ intersects both component boundaries. Moreover, since $[\alpha_k]$ is fixed, such intersection points must be periodic and by considering the denominators of the angles of the rays landing at α_k, we see that these periodic points must have periods dividing 2. However, the relevant periodic rays landing on V_j are given in (2) and those landing on C_j are given in (3). These two sets of angles are separated by the angles $\hat{\beta}_{j-1}$ and $\hat{\beta}_j$, whence the rays landing on α_k (namely the rays of angle θ^1_{k-1} and θ^d_{k-1}) would have to be separated by the rays of angles $-\hat{\beta}_{(j-1)}$ and $-\hat{\beta}_j$, which is a contradiction. $\square$

Now let us consider the ray classes which connect the critical point component on one side with the critical value component on the other.

Lemma 3.8.

- *There exists a ray class $[z]$ of period 2 which contains a point on the boundary of the critical value component of f_j and a point on the critical point component of f_k.*

- *There exists a ray class $[z']$ of period 2 which contains a point on the boundary of the critical value component of f_k and a point on the critical point component of f_j.*

Proof. Observe that $\hat{\beta}_r = -\hat{\beta}_{d-1-r}$ and $\theta^r_{k-1} = -\theta^{d+1-r}_{d-k}$ and so

$$-\theta^r_{j-1} = \theta^{d-r+1}_{d-j-1} \tag{5}$$

In view of 2) and (3) if the two sets

$$\{-\theta^2_{j-1}, -\theta^3_{j-1}, \dots, -\theta^r_{j-1}, \dots, -\theta^{d-1}_{j-1}\}$$

and

$$\{\theta^{d-1}_k, \theta^{d-2}_{k+1}, \dots, \theta^{d-(r-1)}_{k+r-2}, \dots, \theta^2_{k-1+(d-2)}\}$$

intersect, then the intersections would correspond to points where the critical point component of f_k meet the critical value component of f_j at period 2 orbits. By (5) we see that the angle θ^{j+k}_{d-j-1} is the unique element in both these sets and so the associated ray class will provide a periodic point of intersection on the boundaries of the two components. This point is evidently of period 2, and the other member of the period 2 cycle is the ray class that suffices to prove the second claim. □

We refer to the points corresponding to the ray classes of Lemma 3.8 as the periodic meeting points. We need to show that the periodic ray classes found in Lemma 3.8 are in fact the only ray classes which contain a point on the boundary of the critical value component of one polynomial and a point on the boundary of the critical point component of the other.

Proposition 3.9.

- *The critical orbit component of f_j and critical value component of f_k have exactly one boundary point in common under the topological mating.*
- *The critical orbit component of f_k and critical value component of f_j have exactly one boundary point in common under the topological mating.*

In both cases, the meeting point corresponds to the periodic meeting points of Lemma 3.8.

The proof of this proposition will be the goal of the remainder of this section. First we introduce some terminology.

Definition 3.10. A component U of the Fatou set of f_j will be called a level-n component if n is the least natural number such that $f_j^{\circ n}(U)$ is the critical point component. A strictly preperiodic component will be called a primary component if its principal root point lies on the boundary of a periodic Fatou component. A level-n sector is the sector of the sphere which is bounded by the rays landing at the root point of a level-n component and which contains the sector itself and a sector is primary if it is the sector of a primary component (see Figure 3).

Note that by definition the critical point component of f_j is a level-0 component and that the critical value component is a level-1 component. Moreover, if r is the principal root point of a level-n component other than the critical value component, then $f^{\circ n}(r) = \alpha_j$. It is easy to see that all level-1 components are primary, and that their root points lie on the boundary of the critical point component. Also, by Lemma 3.3, each level-1 sector, save for the sector corresponding to the critical value component, contains exactly one ray of angle $\hat{\beta}_r$ for some r. We define the angular width of a sector to be $|\theta_1 - \theta_2|$, where θ_1, θ_2 are the angles of the rays landing at the root of the sector.

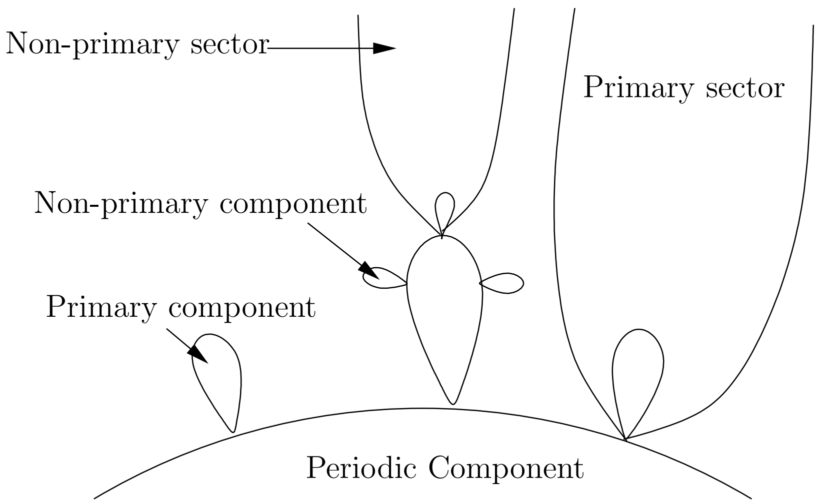

FIGURE 3. The definitions of primary components and sectors.

Let U_1, U_2 be any two primary level-n components for the same polynomial f. U_1 and U_2 both have roots which lie on the boundary of the same critical orbit component C. Removing these two points will split ∂C into two complementary Jordan arcs. We say that the primary level-n component U_1 is *adjacent* (to the periodic meeting point of Lemma 3.8) if there exists a second primary level-n component U_2 such that one of the complementary arcs does not contain a root point of another level-k primary component for all $k \leq n$ but does contain a point which belongs to the ray class which becomes one of the periodic meeting points under the mating. Note that in this case U_2 will also be adjacent. A primary level-n sector will be referred to as being *adjacent* to the periodic meeting point if it corresponds to an adjacent component. Given two adjacent components, there are two rays landing on each of their root points. There is a unique such pair of rays (one landing on the root of each component) such that these rays and ∂C bound a region of $\mathbb{C}$ which does not contain either of the two adjacent components but does contain the periodic meeting point on its boundary. These rays will be called the *closest* rays to the periodic meeting point.

Lemma 3.11. *Suppose U is a primary level-n component. Then*

- *n is even if and only if U shares a boundary with the critical value component.*
- *n is odd if and only if U shares a boundary with the critical point component.*

Proof. Note that if U is not the critical value component, then its root point r is such that $f^{\circ n}(r) = \alpha$. It suffices to prove only the first claim, so assume n is even and, to obtain a contradiction, assume U shares a boundary with the critical point component C. Let r be the root point of U and consider sequences (x_n), $(y_n) \to r$ such that $x_n \in U$ and $y_n \in C$ for all n. Then the sequences $(f^{\circ n}(x_n))$ and $(f^{\circ n}(y_n))$ both converge to α, with both sequences contained in C. This would mean that for a small neighborhood $N \ni r$, the intersection of $f^{\circ n}(N)$ with the critical value component would be empty, which is impossible since $f^{\circ n}$ is locally a homeomorphism at r. □

Given an external ray $R_{f_j}(\theta)$, we will refer to the ray $R_{f_k}(-\theta)$ as its *conjugate ray*. Similarly, the conjugate ray to $R_{f_k}(\theta)$, will be $R_{f_j}(-\theta)$.

Lemma 3.12. *The conjugate rays to those landing on α_j land on the Julia set of f_k in distinct level-1 sector of f_k. Moreover these sectors are adjacent.*

Proof. The rays landing on α_j have angles θ^1_{j-1} and θ^d_{j-1}. If $\theta \in \{\theta^1_{j-1}, \theta^d_{j-1}\}$ then the conjugate ray of angle $-\theta$ lands on a periodic point of period dividing 2 on $J(f_k)$. As the only periodic points of period dividing 2 on the critical point component are α_k and points corresponding to the angles in (3), there exists $n > 1$ such that the ray of angle $-\theta$ lands inside a primary level-n sector S. The root point of S lies on the boundary of the critical point component of f_k, as it is blocked from the critical value component by the rays of angle $-\hat{\beta}_j$ and $-\hat{\beta}_{j-1}$. If $n > 1$, the image of S is a level-$(n-1)$ sector S' which shares a boundary point with the critical value component. However, S' must contain the landing point of the other ray in the 2-cycle, but this is impossible since again this ray is separated from the critical value component by the rays of angles $-\hat{\beta}_j$ and $-\hat{\beta}_{j-1}$. Hence the sector must be a level-1 sector.

By Proposition 3.6, neither of the sectors correspond to the critical value component. If the rays of angles $-\theta^1_{j-1}$ and $-\theta^d_{j-1}$ landed inside the same sector, then that sector would have angular width strictly greater than $1/(d+1)$. However, all level-1 sectors have angular width equal to $1/(d+1)$ and so the two sectors are distinct. Suppose that the two components were not adjacent. By Lemma 3.3, each level-1 sector must contain a ray of angle $\hat{\beta}_r$. However, if the components were not adjacent, then there would exist an angle of the form $-\hat{\beta}_r = -r/(d-1)$ between $-\theta^1_{j-1}$ and $-\theta^d_{j-1}$. This is a contradiction. □

The following lemma will allow us to locate the (primary) preimages of primary components.

Lemma 3.13. *The following hold for the polynomial f_k.*

1. *Given a primary level-n component U which has its root point on the boundary of the critical point component, there exists a unique primary level-$(n+1)$ component U' which is a preimage of U.*
2. *Given a primary level-n component U which has its root point on the boundary of the critical value component, for each $j \neq k$ there exists a unique primary level-$(n+1)$ component U' which is a preimage of U and such that the angles of the rays landing at the root point of U' lie in the arc $(-\theta^d_{j-1}, -\theta^1_{j-1})$.*

Proof.

1. If the level-n component has its root point on boundary of the critical point component, then there exists a unique preimage which has a boundary point on the critical value component, and hence it is primary.
2. If the level-n component has its root point on the boundary of the critical value component, then every preimage component must share a boundary point with the critical point component and so will be primary. The rays of angles $-\theta^d_{j-1}$ and $-\theta^1_{j-1}$ land inside distinct adjacent (primary) level-1 sectors of f_k by Lemma 3.12 and so there exists a unique preimage of U' which have rays landing at its root point whose corresponding angles lie in $(-\theta^d_{j-1}, -\theta^1_{j-1})$.

□

We now use this lemma to study the adjacent components and sectors in the mating $f_j \perp\!\!\!\perp f_k$.

Corollary 3.14. *Consider the unobstructed mating $f_j \perp\!\!\!\perp f_k$. Then for U and U' as in Lemma 3.13, U is adjacent if and only if U' is adjacent.*

Proof. We remark that f_k restricted to the boundary of the critical value component is a homeomorphism and so the proof for U' and U as in the first part of Lemma 3.13 follows.

By Lemma 3.12, the rays of angle $-\theta^d_{j-1}$ and $-\theta^1_{j-1}$ land inside adjacent level-1 sectors. These root points divide the boundary of the critical point component into two Jordan arcs, one of which, A contains the periodic meeting point and no root point of another level-1 component. If U' is adjacent to the periodic meeting point on the critical point component of f_k, then its root point must lie on A. However, f_k restricted to A is a homeomorphism. The existence of this homeomorphism (and its inverse) means that U is adjacent if and only if U' is adjacent in the second case of Lemma 3.13. □

We can now rephrase the previous two results for sectors, to get the following immediate corollary.

Corollary 3.15. *Consider the unobstructed mating $f_j \perp\!\!\!\perp f_k$.*

(1) *Given a primary level-n sector S of f_k which has its root point on the boundary of the critical point component, there exists a unique primary level-$(n+1)$ sector S' which is a preimage of S. Moreover S is adjacent if and only if S' is adjacent.*
(2) *Given a primary level-n sector S which has its root point on the boundary of the critical value component, there exists a unique primary level-$(n+1)$ sector S' which is a preimage of S and such that the angles of the rays landing at the root point of S' lie in the arc $(-\theta^d_{j-1}, -\theta^1_{j-1})$. Moreover S is adjacent if and only if S' is adjacent.*

Lemma 3.16. *Consider the unobstructed mating $f_j \perp\!\!\!\perp f_k$. Let U be a primary level-n component which is adjacent to the periodic meeting point on a critical orbit component of f_k. Then the conjugate ray to the closest ray of U to the periodic meeting point lands inside a level-$(n+1)$ primary sector S of f_j. Furthermore, S is adjacent to the periodic meeting point on the critical orbit component of f_j.*

Proof. We proceed by induction. By Lemma 3.12, the conjugate rays to those landing on α_j land inside the two primary level-1 sectors which are adjacent to the periodic meeting point on the Fatou component of f_k which contains the critical point. There are two rays landing at the root points of each of these sectors, since these points belong to $f_k^{-1}(\alpha_k)$. Consider one of the closest rays to the periodic meeting point. The conjugate ray must land inside a level-2 sector of f_j (by Lemma 3.12 and the fact that the preimage of a level-1 sector is a level-2 sector) and this sector must have its root point on the boundary of a preimage of the critical point component of f_j. It is clear that this component can only be the critical value component of f_j (Figure 4) since all the level-1 (preperiodic) components of f_j will be separated from this ray by the rays landing at α_j, whence this sector is primary. Furthermore, this sector is adjacent by Corollary 3.15.

Now suppose that the statement holds for the two primary level-n components which are adjacent to the periodic meeting point on a critical orbit component of f_k and denote one of them by U'. Take the unique (primary) preimage U of U' that satisfies the conditions of Lemma 3.13 (in the second case, take the preimage for which the rays landing at the root point lie in $(-\theta^d_{j-1}, -\theta^1_{j-1})$). By Corollary 3.14, the preimage U will be an adjacent level-$(n+1)$ component. The conjugate ray to the closest ray landing at the root point of U lands inside a level-$(n+2)$ sector, since it is the pre-image of a level-$(n+1)$ sector. Moreover, by Lemma 3.13 and Corollary 3.15, this sector is adjacent. □

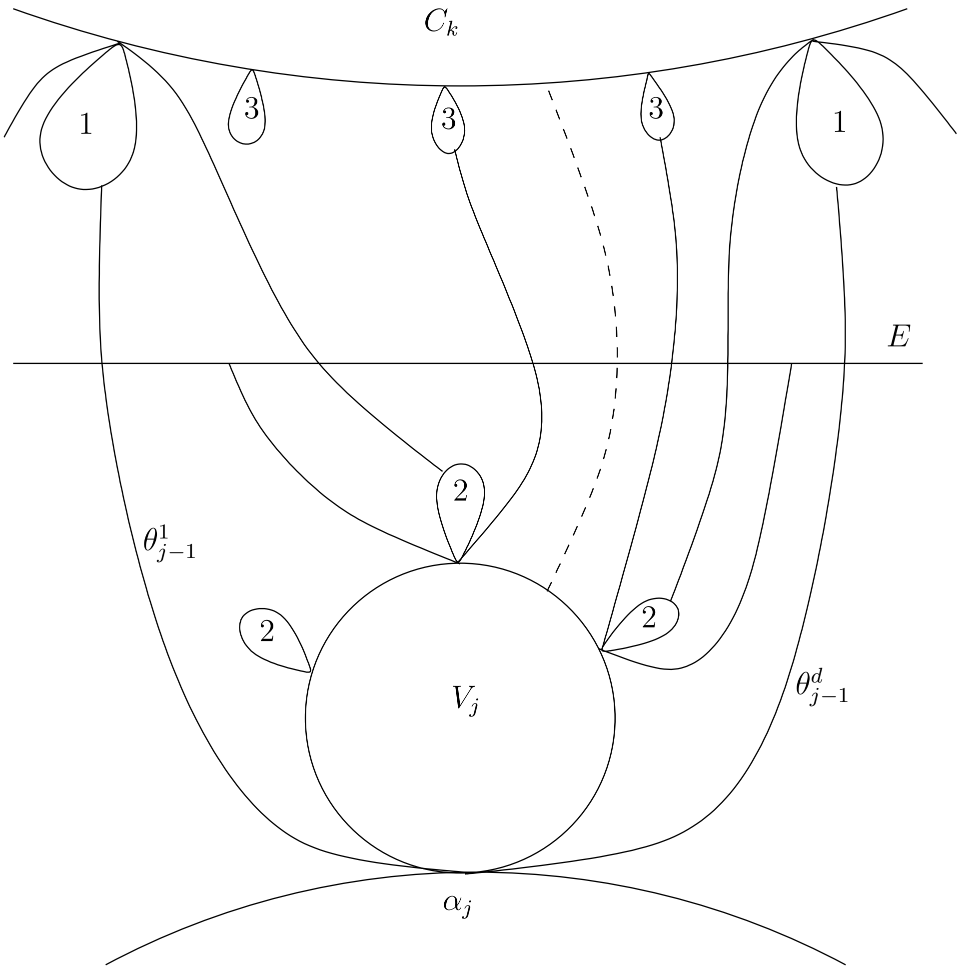

FIGURE 4. The diagram for Lemma 3.16. The dotted line shows the position of the periodic (and by Proposition 3.9, unique) meeting point of the critical value component V_j of f_j and the critical point component C_k of f_k. The numbers inside the components signify their level and E is the equator. Note that any second meeting point would have to correspond to a ray class contained in the region containing the dotted line.

Proof of Proposition 3.9. Again by symmetry it suffices to prove only the first claim. We begin with some notation. We call the *complete ray* of an external ray R to be the union of the ray R along with its landing point, the conjugate ray and the conjugate ray's landing point. Let R_0^1 and R_0^2 be the complete rays landing at α_k. By Lemma 3.12, R_0^1 and R_0^2 have end points lying inside level-1 adjacent sectors of f_j. We denote by R_1^1 and R_1^2 the complete rays which land at the root points of these two level-1 sectors. We inductively define pairs of rays R_n^1 and R_n^2 as follows. Given R_{n-1}^1 and R_{n-1}^2, the complete rays landing at the root points of adjacent level-$(n-1)$ sectors, then in light of Lemma 3.16 define R_n^1 and R_n^2 to be the complete closest rays of the adjacent level-n sectors which include landing points of the complete rays R_{n-1}^1 and R_{n-1}^2; see Figure 4 for the early steps of this construction. Now denote by $\mathcal{C}_n$ the connected component of

$\Sigma \setminus (R_n^1 \cup R_n^2 \cup K(f_j) \cup K(f_k))$ which contains the ray class which collapses to the periodic meeting point p.

If a second meeting point exists, it must correspond to a ray class that is contained in $\mathcal{C}_n$ for all n. To see this, notice that for any point z not contained in some $\overline{\mathcal{C}}_n$, the complete rays R_n^1 and R_n^2 separate z from either V_k, the critical value component of f_k or C_j, the critical point component of f_j, whence z cannot belong to a ray class that becomes a second meeting point. Clearly, $\overline{\mathcal{C}}_{n+1} \subset \overline{\mathcal{C}}_n$ for all n, so we define

$$\mathcal{C} = \bigcap_{i=0}^{\infty} \overline{\mathcal{C}}_n.$$

We claim $\mathcal{C} \cap J(f_k)$ is a singleton $\{p_k\}$, with p_k being the point on the boundary of V_k which becomes the periodic meeting point. Indeed, if $\mathcal{C} \cap J(f_k)$ were not singleton, then it would be an arc in ∂V_k which contains no root points of any primary components. However, there would then exist a neighborhood U of p_k such that $\alpha_k \notin f^{\circ n}(U)$ for all n, which is impossible. Hence $\mathcal{C} \cap J(f_k)$ and similarly, $\mathcal{C} \cap J(f_j)$ are singletons and thus there is only one periodic meeting point. □

3.3. **Structure of the Julia sets of the Rational Maps.** We discuss what the above results tell us about the Julia set of the rational map F. Since the critical value components meet the critical point components at precisely one point (as shown in Lemma 3.7 and Proposition 3.9) , by taking preimages we see that the preimages of the critical point components meet the critical point components at a unique point. Furthermore, the two critical value components and two critical point components don't share boundary points (by Proposition 3.6. Since it must also have rotational symmetry, we see that the Julia set can be described by the schematic as in Figure 5, which shows a period 4 example.

We now have a general picture of what the Julia set of $F \cong f_j \perp\!\!\!\perp f_k$ will look like. An intuitive picture is the following: between the two critical point components (which are neighborhoods of the north and south poles at 0 and ∞) there is a "belt" which contains all the preimages of the critical point components, including the critical value components themselves. Each of these preimage components meets each of the critical point components in precisely one place. We will refer to all maps who have a Julia set with this structure (whether or not they are matings) as type $\mathcal{A}$ maps.

The important piece of data that we require when studying type $\mathcal{A}$ maps will be the displacement δ, which will measure the combinatorial distance between the critical values round the "belt" which forms the space between the two critical point components of the rational map. Let the two critical points be c_1 and c_2 respectively, with respective critical values v_1 and v_2. Form the star which made up of the internal rays in the critical point component of c_1 which go from c_1 to the boundary point where the critical point meets one of the preimages of a critical point component. On the end points of the star thus formed, union the internal rays inside the preimage components which go from the boundary point to the preimage of a critical point contained in the component. Now label the arms which end with a point in the set $F^{-1}(c_2)$ by $\ell_1, \ell_2, \dots, \ell_d$ in clockwise order, where ℓ_1 is the arm which is immediately clockwise of the (unlabeled) arm whose endpoint is v_1.

Definition 3.17. Let F be a rational map of type $\mathcal{A}$. Then the displacement of F will be δ, where ℓ_δ is the arm of the star which has the critical value of the second critical point as an endpoint.

See Figure 5 for an example where the displacement is 3. We note that if F has displacement δ then the map with opposite critical marking F also has displacement δ. We will show that if we know the displacement of a rational map, we actually

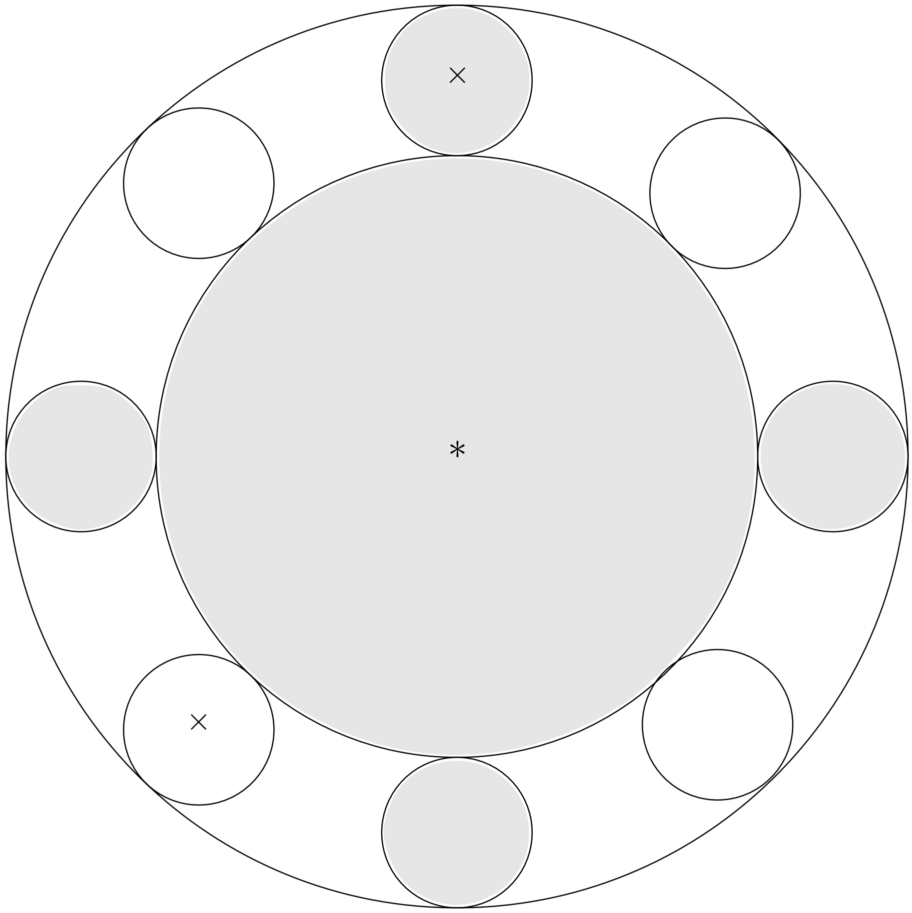

FIGURE 5. The schematic for the Julia set of $F \cong f_j \perp\!\!\!\perp f_k$. The first critical point is labeled with $*$, the second critical point lies outside the larger circle. The two critical values are labeled by $\times$ and here the displacement is 3.

know the rational map up to Möbius conjugacy. Clearly, the displacement is well-defined; the cyclic ordering of the critical value components around the boundary of the critical point components is maintained under Möbius conjugacy. We now show that the two critical values can not be adjacent in the cyclic ordering.

Proposition 3.18. *The displacements $\delta = 1$ and $\delta = d$ are not realized for rational maps.*

We will show that the displacements $\delta = 1$ and $\delta = d$ allow the existence of a Levy cycle. Suppose F is a rational map where the two critical values are adjacent. Denote the Fatou component containing c_1 by C_1. The component C_1 shares a common fixed boundary point ζ with the Fatou component containing $v_1 = F(c_1)$ and has a common period two boundary point ω with the Fatou component containing $v_2 = F(c_2)$. Let λ_1 be the internal ray from v_1 towards ζ with the Fatou component C_1 containing c_1 but perturbed slightly so it does not pass through ζ and ends at a point on ∂C_1 near ζ so that it lies on the arc of $\partial C_1 \setminus \{\zeta, \omega\}$ which contains no boundary points of components which are preimages of the Fatou components containing c_1 or c_2. Let λ_3 be the internal ray from v_2 to ω and λ_2 the arc in ∂C_1 connecting λ_1 and λ_3 such that λ_2 does not contain any boundary points of

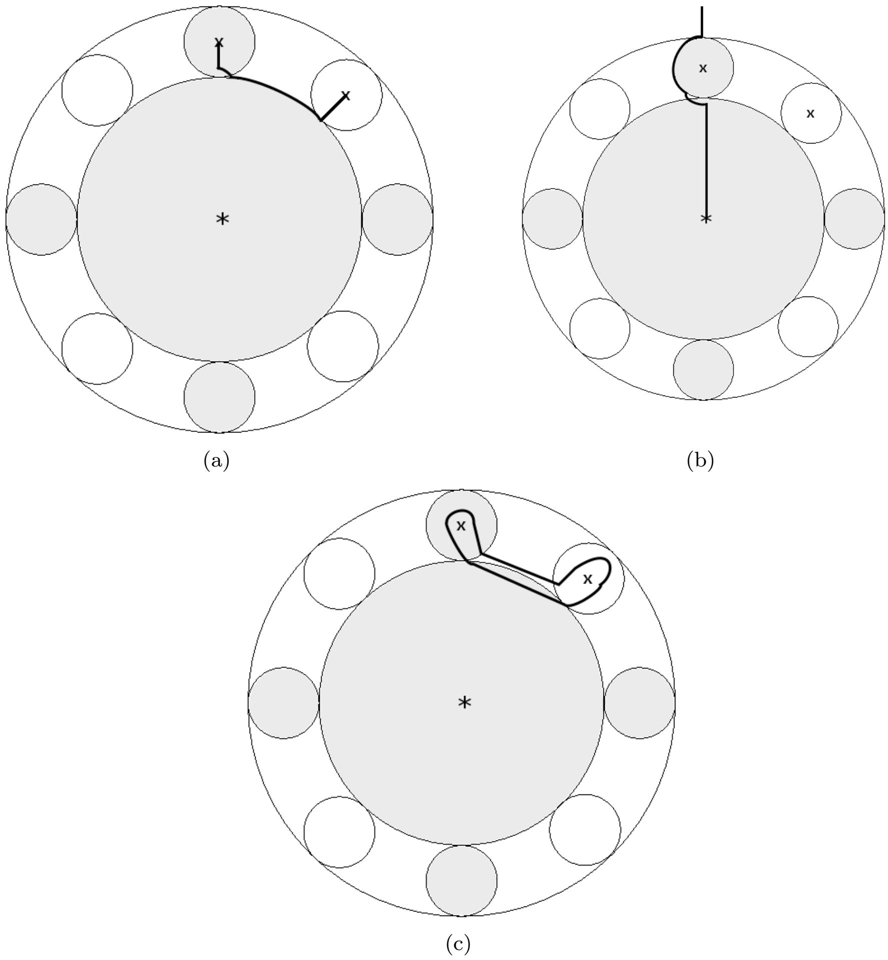

FIGURE 6. The curve Λ is shown in the bold line in (a), a branch of its preimage in (b) and the curve γ (which is a Levy cycle) is shown in (c). Note that the curve in (b) continues to the second critical point, which lies off the diagram.

pre-images of the two critical point components (the existence of λ_2 is guaranteed by the adjacency of the critical value components in the cyclic ordering), see the part (a) of Figure 6. The set $F^{-1}(\Lambda)$ is made up of the union of $d = \deg F$ paths from c_1 to c_2 which go from c_1 along an internal ray towards a preimage of ζ, then pass around this preimage to a point on the boundary of the a Fatou component containing an element of $F^{-1}(c_1)$. It then travels along the boundary of this component to the element in $F^{-1}(\omega)$ which is also on the boundary. It then travels along an internal ray to c_2, see part (b) of Figure 6 for an example of one of these preimages. It is clear that Λ and the set $F^{-1}(\Lambda)$ have an empty intersection.

Proof of Proposition 3.18. We use the Λ from the previous paragraph to help us construct a multicurve Γ which we will show is a Levy cycle. Let γ be the boundary of an ϵ-neighborhood of Λ so that the component of $\widehat{\mathbb{C}} \setminus \gamma$ containing the two critical values does not contain ζ, nor either of the critical points, see part (c) of Figure 6. By considering the details of $F^{-1}(\Lambda)$, we then see that $F^{-1}(\gamma)$ is made up of $d-1$ peripheral curves and one curve γ' which is homotopic to γ relative to P_F (but with the orientation reversed). We then see that $F_{|\gamma'} : \gamma' \to \gamma$ is a homeomorphism and so $\Gamma = \{\gamma\}$ is a Levy cycle. □

In the following proposition, we use $d-1$ as the representative of the integers equal to 0 modulo $d-1$.

Proposition 3.19. *The degree d rational map $F \cong f_j \perp\!\!\!\perp f_k$ has displacement $\delta \equiv j+k \mod (d-1)$.*

Proof. The displacement is equal to the number of preimage components of the critical point component of f_j between the critical value component of f_j and the critical value component of f_k (inclusive of the first component). By the result of Lemma 3.3, this is equal to the number of angles of the form $\hat{\beta}_r$ between $\hat{\beta}_{j-1}$ (inclusive) and $-\hat{\beta}_k = \hat{\beta}_{d-k-1}$. This value is clearly $j+k$. □

Remark 3.20. Notice that, using the convention above, the two non-realizable cases $\delta = 1$ and $\delta = d$ both correspond to $j+k \equiv 1 \mod d-1$.

Corollary 3.21. *Each displacement can be obtained by matings in precisely $d-1$ ways.*

Proof. Suppose we want to get the displacement to be δ. Given any f_j, the rational map $F \cong f_j \perp\!\!\!\perp f_{\delta-j}$ has displacement δ. The map $f_{\delta-j}$ exists and is unique for each j, and since there are $d-1$ choices for j, the result follows. □

Proof of Theorem A. By the algebraic classification in Section 2 there are exactly $d-2$ degree d rational maps (up to Möbius conjugacy) with two period two superattracting cycles. Furthermore, by Proposition 3.19, we can obtain $d-2$ different displacements, each of which must correspond to a different rational map and by Corollary 3.21 each displacement is obtained in precisely $d-1$ ways. □

4. Thurston Classification

We now show that the critical displacement is enough to classify a rational map in the sense of Thurston. The advantage of this approach is that, in general, "clustering" (as defined in [Sha12a, Sha11, Sha12b]) is not an algebraic condition, and therefore the approach used above, by using the algebraic normal form of Section 2, will not work in the general case where the period of the critical orbits is greater than 2. In this section we will show how the result of Theorem A can be obtained without reference to this algebraic classification.

Theorem 4.1. *Suppose F and G are bicritical rational maps of type $\mathcal{A}$ and have the same displacement δ. Then F and G are equivalent in the sense of Thurston.*

Recall that to satisfy the conditions of Thurston equivalence, we need to find two homeomorphisms Φ and $\widehat{\Phi}$ with the following properties:

(1) $\Phi \circ F = G \circ \widehat{\Phi}$.
(2) $\Phi|_{P_F} = \widehat{\Phi}|_{P_F}$.
(3) Φ and $\widehat{\Phi}$ are isotopic rel P_F.

We will first of all construct the homeomorphism Φ, making use of the Alexander trick. We will then define the homeomorphism $\widehat{\Phi}$, making sure that each condition above in turn is satisfied. Since we are considering maps of type $\mathcal{A}$, we can take advantage of what we know about the Julia sets of these maps. Denote the critical points of F by c_1 and c_2, with values v_1 and v_2, and the critical points of G by c_1' and c_2', with values v_1' and v_2'. Construct a closed curve γ in the F-sphere as follows. Let γ_1 be the path through internal rays from c_1 to v_1, γ_2 the path through internal rays from v_1 to c_2, γ_3 the path through internal rays from c_2 to v_2 and γ_4

the path through internal rays from v_2 to c_1. Define the curve γ' in the G-sphere in the analogous way.

We observe that $F\colon \gamma \to \gamma$ and $G\colon \gamma' \to \gamma'$ are homeomorphisms. It is easy to see that there exists $\phi\colon \gamma \to \gamma'$ which conjugates the dynamics of F and G on γ and γ' respectively. Then, by two applications of the Alexander trick, we can extend ϕ to a homeomorphism $\Phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Notice that the set $F^{-1}(\gamma)$ divides the F-sphere up into $2d$ regions and similarly, the set $G^{-1}(\gamma')$ divides the G-sphere up into $2d$ regions. Label the regions in the F-sphere $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{2d}$, starting with the region immediately anticlockwise of the curve γ at c_1. Similarly label the regions in the G-sphere by $\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_{2d}$.

Lemma 4.2. *There exists a homeomorphism* $\widehat{\Phi}$ *so that* $\Phi \circ F = G \circ \widehat{\Phi}$ *and* $\Phi|_\gamma = \widehat{\Phi}|_\gamma$.

Proof. We now use this labeling to define the second homeomorphism $\widehat{\Phi}$. To satisfy the Thurston criterion, we require $\Phi \circ F = G \circ \widehat{\Phi}$, so that for each z we have $\widehat{\Phi}(z) \in G^{-1}(\Phi(F(z)))$. If $z \in \mathcal{A}_i$, then let $\widehat{\Phi}(z)$ be the unique point in $G^{-1}(\Phi(F(z))) \cap \mathcal{A}'_i$. We can extend the map $\widehat{\Phi}|_{\mathcal{A}_i}$ continuously to the boundary for each i, and in particular, since the displacements of F and G are the same, we have $\Phi|_\gamma = \widehat{\Phi}|_\gamma$. Finally, $\widehat{\Phi}$ is a homeomorphism since it is continuous and the sphere is a compact, Hausdorff space. □

Lemma 4.3. $\widehat{\Phi}$ *is isotopic to* Φ *rel* γ.

Proof. We know that $\Phi|_\gamma = \phi = \widehat{\Phi}|_{\gamma'}$. The complement to γ is two topological discs and so by the Alexander trick, Φ and $\widehat{\Phi}$ are isotopic to each other on both of these discs rel the boundary. Hence they are isotopic rel γ. □

Proof of Theorem 4.1. By Lemma 4.2, we see that $\Phi \circ F = G \circ \widehat{\Phi}$ and $\Phi|_\gamma = \widehat{\Phi}|_\gamma$, and hence $\Phi|_{P_F} = \widehat{\Phi}|_{P_F}$. Then, by Lemma 4.3, the homeomorphisms Φ and $\widehat{\Phi}$ are isotopic rel γ and hence rel P_F. Thus F and G are equivalent. □

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