RAMSEY- AND TURÁN-TYPE PROBLEMS

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RAMSEY THEORY

OF THREE ORDINARY PEOPLE,
TWO MUST HAVE THE
SAME SEX

D.J. KLEITMAN

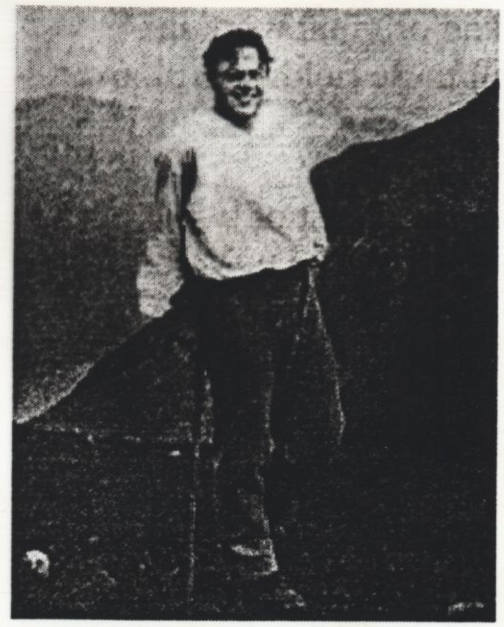
RAMSEY THEORY REFERS TO A LARGE BODY OF DEEP RESULTS IN MATHEMATICS WITH UNDERLYING PHILOSOPHY

IN LARGE SYSTEM COMPLETE DISORDER IS IMPOSSIBLE

RAMSEY'S THEOREM

FRANK PLUMPTON
RAMSEY

1903 – 1930



THEOREM: $\forall\, k, \ell \;\; \exists\, N$ s.t.

ANY TWO-COLORING OF THE EDGES OF COMPLETE GRAPH ON N VERTICES CONTAINS EITHER

- RED COMPLETE GRAPH OF SIZE k

OR

- GREEN COMPLETE GRAPH OF SIZE ℓ

BOUNDS ON RAMSEY NUMBERS

DEF: $R(k,\ell) = \text{MIN. } N(k,\ell)$ s.t. IF THE EDGES OF COMPLETE GRAPH K_N ARE COLORED RED AND GREEN THEN EITHER THERE IS

- RED COPY OF K_k OR
- GREEN COPY OF K_ℓ

THEOREM: [ERDŐS SZEKERES]

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}$$

DIAGONAL CASE

$$R(k,k) \leq \binom{2k-2}{k-1} \approx 2^{2k}$$

ERDŐS EXISTENCE ARGUMENT

THEOREM: $R(k,k) \geq 2^{\frac{k}{2}}$

[ERDŐS 1947]

PROOF: COLOR THE EDGES OF COMPLETE GRAPH K_N, $N = 2^{\frac{k}{2}}$ RED AND GREEN RANDOMLY AND INDEPENDENTLY WITH PROBABILITY $1/2$

THE PROBABILITY THAT THERE IS MONOCHROMATIC COPY OF K_k

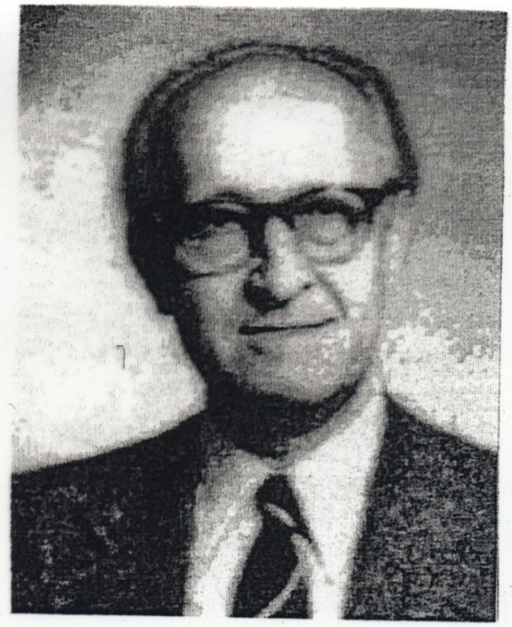
$$\leq \binom{N}{k} \frac{1}{2^{\binom{k}{2}-1}} << 1$$

$\blacksquare$

TURÁN NUMBERS

PAUL TURÁN

1910 - 1976



DEF: FOR A FIXED GRAPH H

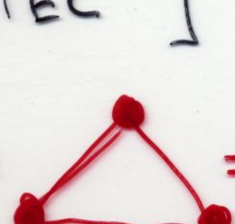
$ex(n, H) =$ max # OF EDGES IN G
s.t. G HAS n VERTICES
AND NO COPY OF H

TM:

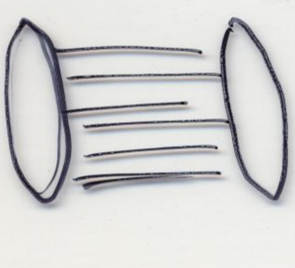
K_k = COMLETE GRAPH OF ORDER k

[TURÁN 41
FOR k=3
MANTEL]

$ex(n, K_k) =$ # OF EDGES IN $k-1$ PARTITE GRAPH ON n VERTICES WITH ALMOST EQUAL PARTS

$H =$  $= K_3$

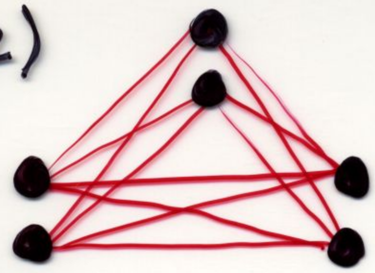
$$= \left(1 - \frac{1}{k-1}\right)\frac{n^2}{2} + O(1)$$



ERDŐS - STONE THEOREM

DEF: $K_k(t)$ = COMPLETE k-PARTITE GRAPH WITH EVERY PART OF SIZE t

$K_3(2)$



TH [ERDŐS-STONE 1946]

$$ex(n, K_k(t)) \leq \left(1 - \frac{1}{k-1} + o(1)\right)\frac{n^2}{2}$$

FOR ANY FIXED t

[VARIOUS EXTENSIONS : BOLLOBÁS, ERDŐS, SIMONOVITS]

COROL: [ERDŐS-SIMONOVITS] H FIXED GRAPH, $\chi(H) = k \geq 3$

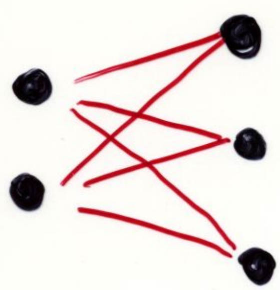
$$ex(n, H) = \left(1 - \frac{1}{k-1} + o(1)\right)\frac{n^2}{2}$$

BIPARTITE TURÁN NUMBERS

DEF: $K_{t,s}$ = COMPLETE BIPARTITE GRAPH WITH PARTS OF SIZE S AND t

C_{2K} = CYCLE OF LENGTH 2K

$K_{2,3}$



C_6

TH: [KŐVARI - SÓS - TURÁN 54]

LET $s \geq t$ THEN

$$ex(n, K_{t,s}) \leq O\left(n^{2-1/t}\right)$$

TH: [BONDY - SIMONOVITS 1974]

$$ex(n, C_{2K}) \leq O\left(n^{1+1/k}\right)$$

CONJECTURE

$$ex(n, K_{t,s}) = \Theta(n^{2-1/t}) \quad s \geq t$$

$$ex(n, C_{2k}) = \Theta(n^{1+1/k})$$

KNOWN RESULTS

ERDŐS - RÉNYI - SÓS 1966

$$ex(n, K_{2,2}) = ex(n, C_4) = \Omega(n^{3/2})$$

BROWN 1966

$$ex(n, K_{3,3}) = \Omega(n^{5/3})$$

ALON, RÓNYAI, SZABÓ 99 EXTENDING RESULTS OF KOLLÁR, RÓNYAI, SZABÓ 96

FOR $s \geq (t-1)! + 1$

$$ex(n, K_{t,s}) \geq \Omega(n^{2-1/t})$$

BENSON 1966

$$ex(n, C_6) = \Omega(n^{4/3})$$

$$ex(n, C_{10}) = \Omega(n^{6/5})$$

COMMON DENOMINATOR OF BOTH PROBLEMS

GIVEN TWO GRAPHS G AND H SHOW THAT

G CONTAINS COPY OF H !

OFTEN:

- H HAS FIXED SIZE OR IS SPARSE
- G IS SUFFICIENTLY DENSE

NOTE: ONE OF THE COLORS CONTAINS AT LEAST $\frac{1}{2}\binom{N}{2}$ EDGES

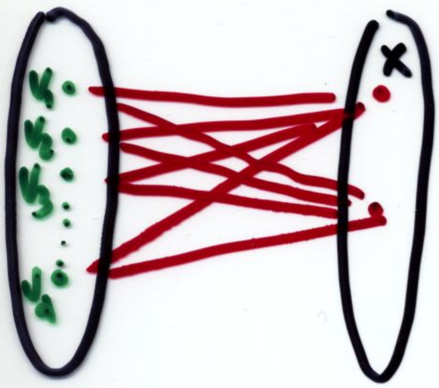
QUESTION:

WHICH CONFIGURATIONS CAN WE FIND IN THE LARGE, DENSE (DENSITY $\geq \frac{1}{2}$) GRAPHS?

USEFUL CONFIGURATION

LET G BE A GRAPH, $|V(G)| = n$

WE NEED: "BIG SET" $U \subseteq V(G)$



U

$N(v_1, v_2, \dots, v_d)$

$\forall\ v_1, \dots, v_d$ d VERTICES IN U HAVE "MANY" COMMON NEIGHBOURS

NOTE: ON AVERAGE EVERY d VERTICES OF G WITH $\geq \frac{1}{2}\binom{n}{2}$ EDGES HAVE

$$\geq \frac{n\binom{n/2}{d}}{\binom{n}{d}} \gtrsim \frac{n}{2^d}$$ COMMON NEIGHBOURS

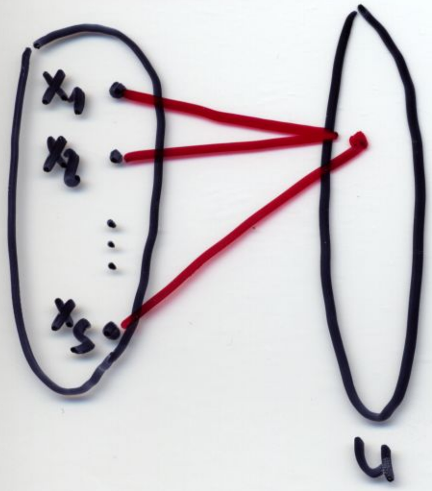
LEMMA:

[RÖDL, KOSTOCHKA, GOWERS, 2 S.]

LET G BE A GRAPH ON n VERTICES WITH $\geq \frac{1}{2}\binom{n}{2}$ EDGES

THEN $\exists$ SUBSET OF VERTICES U, $|U| \geq \sqrt{n}$ s.t. ANY d = CONSTANT VERTICES IN U HAVE AT LEAST $\geq \frac{n}{2^{2d}}$ COMMON NIGHBOURS

PROOF: CHOOSE $s = \frac{1}{2}\log_2 n$ RANDOM VERTICES OF G. LET U BE THE SET OF COMMON NIGHBOURS OF THESE VERTICES



EXPECTED SIZE OF U

$$E(|U|) = \sum_{v \in G} \left(\frac{d(v)}{n}\right)^s \geq n \cdot \left(\frac{n/2}{n}\right)^s =$$

$$= n \left(\frac{1}{2}\right)^{\frac{1}{2}\log_2 n} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

[NOTE ALL DEGREES $\approx n/2$]

EXPECTED # OF d-TUPLES OF VERTICES IN U WITH $< n/2^{2d}$ COMMON NIGHBOURS

$$E\begin{pmatrix}\#\text{ of}\\ d\text{-tuples}\end{pmatrix} \leq \binom{n}{d} \left(\frac{n/2^{2d}}{n}\right)^s \leq$$

$$< n^d \left(\frac{1}{2^{2d}}\right)^{1/2 \log_2 n} = n^d \cdot \frac{1}{n^d} = 1$$

DELETE ONE VERTEX FROM EVERY SUCH d-TUPLE

GRAPH RAMSEY THEORY

DEF: GIVEN A GRAPH G

$R(G) =$ MIN N s.t. IN ANY TWO-COLORING OF COMPLETE GRAPH K_N THERE IS MONOCHROM-ATIC COPY OF G

PROBLEM: BEHAVIOUR OF $R(G)$ WHEN G IS RELATIVELY SPARSE $\approx$

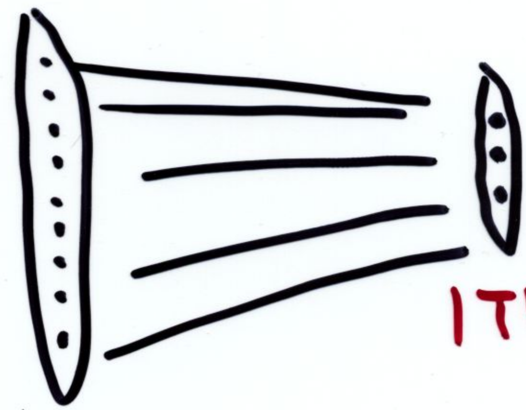
- FEW EDGES
- NO LARGE COMPLETE SUBGRAPHS

CONJECTURE OF BURR - ERDŐS

DEF: GRAPH G IS CALLED d-DEGENERATE IF EVERY SUBGRAPH OF G HAS VERTEX OF DEGREE $\leq d$.

EXAMPLE:

THIS COMPLETE BIPARTITE GRAPH IS 3-DEGENERATE



$|T| = 3$

$|S| = n$

CONJECTURE [BURR & ERDŐS 1975]

$\forall d \;\; \exists \; c(d)$ s.t. $\forall$ d-DEGENERATE GRAPH G ON n VERTICES

$$R(G) \leq c(d) \cdot n$$

KNOWN RESULTS

THEOREM: [CHVATAL, RÖDL, SZEMERÉDI & TROTTER 1983] $\forall d$ $\exists c(d)$ SUCH THAT $\forall$ GRAPH G WITH MAXIMUM DEGREE d AND n VERTICES

HOLDS $R(G) \leq c(d) \cdot n$

BETTER BOUNDS ON $c(d)$: EATON; GRAHAM, RÖDL & RUCIŃSKI

SPECIAL CASES: ALON; CHEN & SCHELP; RÖDL & THOMAS

THEOREM: [KOSTOCHKA & RÖDL 2001] $\forall d$ $\exists c(d)$ SUCH THAT $\forall$ d-DEGENERATE GRAPH G ON n VERTICES

$$R(G) \leq c(d) \cdot n^2$$

NEW RESULT

THEOREM: $\forall \varepsilon > 0 \ \exists n_0(\varepsilon)$ s.t.
[KOSTOCHKA & S. 2002] $\forall$ d-DEGENERATE GRAPH G ON $n > n_0(\varepsilon)$ VERTICES

HOLDS $$R(G) \leq n^{1+\varepsilon}$$

MOREOVER:

- d CAN SLIGHTLY GROW WITH n
- IF G IS BIPARTITE THEN THE RESULT STILL HOLDS FOR $d = o(\log n)$ OR

IF WE USE MORE THAN TWO COLORS

BIPARTITE r-DEGENERATE GRAPHS

DEF: GRAPH IS r-DEGENERATE IF EVERY SUBGRAPH OF IT CONTAINS A VERTEX OF DEGREE $\leq r$

CONJECTURE:

[Erdős AND SIMONOVITS 1967]

$$ex(n,H) \leq O(n^{2-\frac{1}{r}})$$

FOR EVERY FIXED BIPARTITE r-DEGENERATE GRAPH H

[THIS IS ROUGHLY $ex(n, K_{r,r})$]

SOME NEW BOUNDS

TH 1: LET H BE A BIPARTITE GRAPH WITH MAXIMUM DEGREE r ON ONE SIDE

[ALON, KRIVELEVICH & S.]

THEN $$ex(n, H) \le c\, n^{2-\frac{1}{r}}$$

(THIS IS TIGHT, E.G., FOR $K_{r,s}$ $s \ge (r-1)!+1$)

CAN ALSO BE DERIVED FROM FÜREDI (91)

TH 2: FOR EVERY r-DEGENERATE BIPARTITE GRAPH H AND FOR EVERY $n \ge n_0(H)$:

[ALON, KRIVELEVICH & S.]

$$ex(n, H) \le n^{2-\frac{1}{10r}}$$

GRAPHS WITH FIXED NUMBER OF EDGES

CONJECTURE [ERDŐS 1983] LET G BE A GRAPH WITH m EDGES, THEN

$$R(G) \le 2^{c\sqrt{m}}$$

FOR SOME CONSTANT $c>0$.

EXAMPLES WHEN THIS IS TIGHT

$G =$ 

COMPLETE GRAPH OF ORDER $\sqrt{2m}$

$G =$ 

COMPLETE BIPARTITE GRAPH WITH PARTS OF SIZE $\sqrt{m}$

BIPARTITE CASE

THEOREM: LET G BE A BIPARTITE GRAPH WITH m EDGES

[ALON, KRIVELEVICH AND S.]

THEN
$$R(G) \leq 2^{c\sqrt{m}}$$

[THIS IS TIGHT AND PROVES CONJECTURE FOR BIPARTITE CASE]

MOREOVER FOR **EVERY** GRAPH G WITH m EDGES
$$R(G) \leq 2^{c'\sqrt{m}\log m}$$

ADDITIONAL APPLICATIONS OF LEMMA AND ITS VARIANTS

RÖDL
KOSTOCHKA

QUESTION OF TROTTER ON RAMSEY NUMBERS OF BIPARTITE GRAPHS

GOWERS

NEW PROOF OF BALOG-SZEMERÉDI THEOREM

B. S.

QUESTIONS FROM RAMSEY-TURÁN THEORY AND BOUNDS ON GENERALIZED RAMSEY NUMBERS

ALON, KRIVELEVICH AND S.

QUESTION OF ERDŐS-McKAY ON THE SIZES OF INDUCED SUBGRAPHS OF RAMSEY GRAPH

ALON KRIVELEVICH AND S.

TIGHT BOUNDS ON TURÁN NUMBERS OF TWO LEVELS OF THE BOOLEAN LATTICE

MORE TO COME (?)