A Survey of Lattice Path Enumeration

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①

Bertrand's ballot problem (1887)

In an election with two candidates the winner receives m votes and the loser n votes. What is the probability that the winner is always ahead?

We can represent an arrangement of the ballots by a path in the plane

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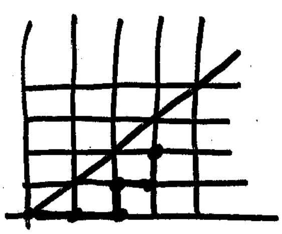

Path stays below $x=y$

Path stays above x-axis

Note that this is really a one-dimensional problem

Answer: The probability is $\frac{m-n}{m+n}$. The number of paths is $\frac{m-n}{m+n}\binom{m+n}{m} = \binom{m+n-1}{m-1} - \binom{m+n-1}{m}$

③

Elegant proof by D. André in 1887, using the reflection principle.

We count paths from $(1,0)$ to (m,n) that never touch $x=y$ (good paths).

$$\#\text{good paths} = \#\text{all paths } \binom{m+n-1}{m-1} - \#\text{bad paths}$$

We show

$$\#\text{bad paths } (1,0) \text{ to } (m,n) = \#\text{all paths } (0,1) \text{ to } (m,n) = \binom{m+n-1}{m}$$

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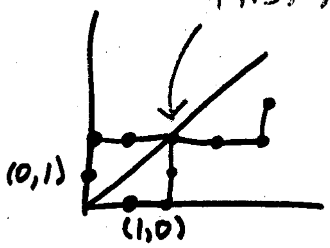
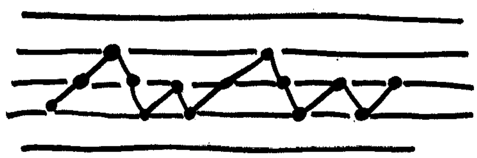


André's bijection

Another application of the reflection principle:
Count paths that stay strictly between two parallel lines.



(H. Grossman, 1949, "Fun With Lattice Points")

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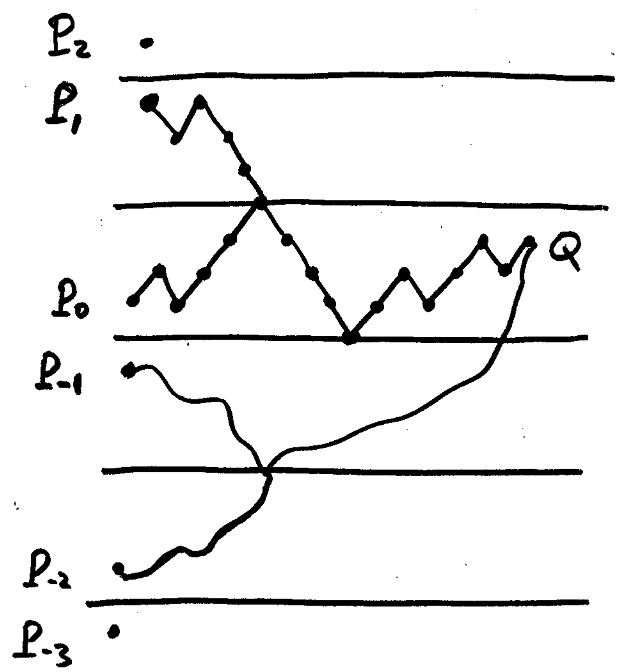
$$\#\text{good paths} = N(P_0,Q) - N(P_1,Q) - N(P_{-1},Q) + N(P_2,Q) + N(P_{-2},Q) - \cdots$$



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The reflection principle can be applied much more generally:

We take a set of hyperplanes closed under reflection, such that lattice points are reflected to lattice points. Then we can count paths restricted to regions bounded by the hyperplanes. (Groups generated by reflections are Weyl groups or affine Weyl groups.)

G. & Zeilberger, Magyar & Grabiner, Krattenthaler

Important special case: standard Young tableaux (multi-candidate ballot problem)

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The method of images

(J. Peacock, On "Al Capone and the Death Ray" 1942)

Analogous to Kelvin's method of images in electrostatics:

To solve a $\left\{\begin{matrix}\text{differential}\\ \text{difference}\end{matrix}\right\}$ equation with boundary conditions, we take linear combinations of known solutions to satisfy the boundary conditions.

(8)

The ballot problem:

Let $B(m,n)$ be the number of paths from $(1,0)$ to (m,n), $m>n$, that stay below the diagonal. Then

$$B(m,n)-B(m-1,n)-B(m,n-1) = \begin{cases} 0, & \text{if } m>n,\ (m,n)\neq(1,0) \\ 1, & \text{if } (m,n)=(1,0) \end{cases}$$

$B(m,n)=0$ if $n<0$ or $m=n$

(9)

Let $C(m,n) = \dfrac{(m+n)!}{m!\,n!}$

Then $C(m,n)$ satisfies the same recurrence. More generally, let

$$f(m,n) = C(m-a, n-b)$$

Then

$$f(m,n)-f(m-1,n)-f(m,n-1) = \begin{cases} 0 & \text{if } (m,n)\neq(a,b) \\ 1 & \text{if } (m,n)=(a,b) \end{cases}$$

We want a linear combination of the $f(m,n)$ that satisfies the boundary condition for $B(m,n)$

(10)

Construct a table of $B(m,n)$ for $m\geq n$

-1	-3	-5	-5	0
-1	-2	-2	0	5
-1	-1	0	2	5
-1	0	1	2	3
0	1	1	1	1

Extend to $m<n$ so that the recurrence is still satisfied everywhere except at $(1,0)$ and $(0,1)$

This suggests that
$B(m,n) = C(m-1,n) - C(m,n-1)$
and the boundary condition is easily verified.

(11)

Unlike the reflection principle, the method of images doesn't require symmetry.

Let $T(m,n)$ be the number of paths from $(1,0)$ to (m,n) that stay below $x=2y$.

-2	-5	-8	-10	-10	-7	0
-2	-3	-3	-2	0	3	7
-2	-1	0	1	2	3	4
0	1	1	1	1	1	1

This suggests that
$T(m,n) = C(m-1,n) - 2C(m,n-1)$

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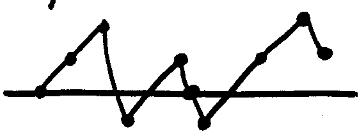
Factorization methods

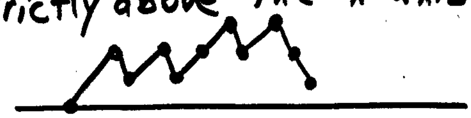
(One-dimensional paths)

Think of a path as a sequence of steps. We can concatenate paths and factor them. This gives formulas for their generating functions.

Draw paths like this:



Call a path positive if it stays strictly above the x-axis



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Simplest case:
Steps that go up and down by 1. 

Let P_k be the set of positive paths ending at height k.
Then every path in P_k can be factored uniquely into k paths in P_1.

Weight each step by x and let f_k be the generating function for paths in P_k with $f_1 = f$.
Then
$$f_k = f^k$$

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If we remove the last step from a path in P_1 we get either an empty path

 or a path in P_2 

So $f = x(1+f^2)$. We can solve this quadratic equation to get the generating function for Catalan numbers.

The same reasoning applies to paths with arbitrary steps as long as we can't go up by more than 1.

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If we weight a step that goes down i by $x g_{i+1}$ (including $i=0$ and $i=-1$) then
$$f(x) = x\sum_{i=-1}^{\infty} g_{i+1} f(x)^{i+1}$$
$$= x\sum_{j=0}^{\infty} g_j f(x)^j$$

How can we solve this?

Lagrange inversion

If $f(x) = x\,G(f(x))$ then
$$[x^n]\, f(x)^k = \frac{k}{n}[t^{n-k}]\, G(t)^n$$

(16)

But we can count these paths directly, getting a combinatorial proof of Lagrange inversion, using the Cycle Lemma (Dvoretzky-Motzkin, Raney)

For any path with steps that go down by $-1, 0, 1, 2, \cdots$, ending at height $k>0$, there are exactly k cyclic shifts that are positive paths.

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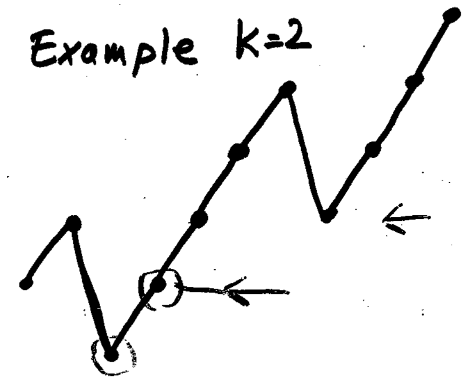
Example $k=2$



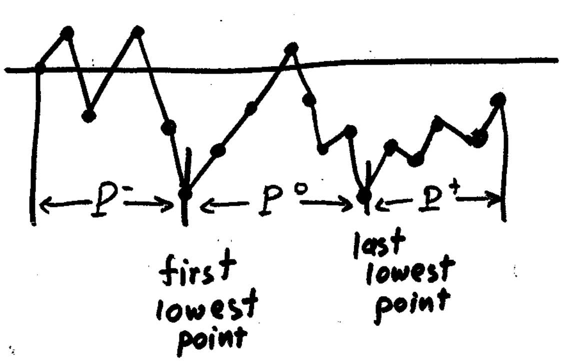
Thus among all paths of length n, ending at height k, and consisting of a given collection of steps, a proportion $\frac{k}{n}$ are positive.

Example (Catalan numbers)
$n+1$ steps up, n steps down.
There are $\binom{2n+1}{n}$ paths in all;
$\frac{1}{2n+1}\binom{2n+1}{n}$ are positive

(18)

Another factorization method (Wiener-Hopf) that applies to an arbitrary set of integer steps.

$$P = P^- P^0 P^+$$



P^+ is a positive path.
P^0 is nonnegative path ending at height 0 (zero path)
P^- is a reversed positive path (negative path)

(19)

So for a fixed, but arbitrary, set of steps,

f = g.f. for all paths
= (g.f. for negative paths)
· (g.f. for zero paths)
· (g.f. for positive paths)
$= f_- f_0 f_+$

Surprising fact: f_-, f_0, and f_+ are uniquely determined by f (as long as the g.f.s keep track of endpoints)

$$\log f = \log f_- + \log f_0 + \log f_+$$

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A probabilistic method

Applies to a wide class of problems, not necessarily 1-dimensional.

Gives a functional equation (but different from factorization).

Related to the "kernel method."

Simplest example:

Fix k. Let r_n be the number of paths from $(0,0)$ to $(n, n+k)$ that touch the line $y=x+k$ for the first time at the end. (The ballot problem again.)

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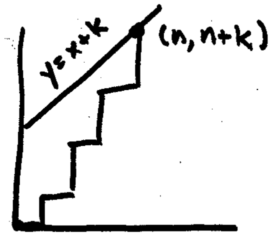

Let $R(x) = \sum_{n=0}^{\infty} r_n x^n$.

Consider a particle that starts at $(0,0)$ and moves right with probability p and up with probability $q=1-p$. If it hits the line $y=x+k$, it stops.

The probability that it hits $(n,n+k)$ (and stops there) is $r_n p^n q^{n+k}$

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So the probability that it stops is

$$\sum_{n=0}^{\infty} r_n p^n q^{n+k} = q^k R(pq).$$

But if p is small $(\leq \frac{1}{2})$ then it stops with probability 1.

So
$$q^k R(pq) = 1$$
$$R(pq) = \frac{1}{q^k}$$
$$R(p(1-p)) = \frac{1}{(1-p)^k}$$

To solve this, we set $p(1-p)=x$ and solve for p:

$$p = \frac{1-\sqrt{1-4x}}{2} \qquad R(x) = \left(\frac{1-\sqrt{1-4x}}{2x}\right)^k$$

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A conjecture

Let $P(m)$ be the number of paths of length $2m$ in the plane, starting and ending at $(0,1)$, with unit steps north, south, east, and west, and staying in the region $y>0$ $x>-y$.

Then
$$P(m) = 16^n \frac{(\frac{1}{2})_m (\frac{5}{6})_m}{(2)_m (\frac{5}{3})_m}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$