

MTH141 Stewart Section 4.5

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Indeterminate Forms and l'Hospital's Rule

Sometimes it is necessary to determine the limit of a function at a point where the function is not defined, for example

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$$

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We've seen that limits of rational functions like this one can be handled by factoring because the numerator and denominator have a common factor ($x - 2$ in this case):

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{1}{x + 2} = \frac{1}{4}$$

Indeterminate Forms and l'Hospital's Rule

Although we cannot evaluate

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at $x = 2$, we can evaluate the numerator and denominator separately:

$$(x - 2) = 0 \quad \text{when} \quad x = 2$$

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We cannot evaluate the fraction, because when $x = 2$, it has the form

$$\frac{0}{0}$$

which is called an **indeterminate form of type** $\frac{0}{0}$.

Indeterminate Forms and l'Hospital's Rule

Suppose we label the functions in the numerator and denominator as f and g , respectively:

$$f(x) = x - 2$$

$$g(x) = x^2 - 4$$

Indeterminate Forms and l'Hospital's Rule

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$$\begin{aligned}f(x) &= x - 2 \\g(x) &= x^2 - 4\end{aligned}$$

Then the derivatives of f and g are:

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x - 2) = 1 \\g'(x) &= \frac{d}{dx}(x^2 - 4) = 2x\end{aligned}$$

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Notice that $f'(2) = 1$ and $g'(2) = 2 \cdot 2 = 4$. Since $g'(2) \neq 0$, the rational function

$$\frac{f'(x)}{g'(x)}$$

is perfectly well-defined at $x = 2$.

Indeterminate Forms and l'Hospital's Rule

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We can find the limits of the numerator and denominator as $x \rightarrow 2$:

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Since $g'(2) \neq 0$ we can use direct substitution to obtain

$$\lim_{x \rightarrow 2} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} 2x} = \frac{1}{4}$$

Indeterminate Forms and l'Hospital's Rule

Notice that from the previous result

$$\lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \frac{1}{4}$$

so as it turns out

$$\lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 2} \frac{f'(x)}{g'(x)} = \frac{1}{4}$$

This is not a coincidence, but an example of a theorem called l'Hospital's Rule.

Indeterminate Forms and l'Hospital's Rule

Theorem(l'Hospital's Rule):

Suppose f and g are differentiable and $g'(x) \neq 0$ for values of x near $x = a$ (with the possible exception of a itself).

Suppose also that the limits as $x \rightarrow a$ of f and g are *both* zero or *both* $\pm\infty$.

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$$

or

$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, or is $\pm\infty$.