MTH 535 Fall 2004 An application of Fubini's Theorem

Theorem Let $P(x_1, \ldots, x_n)$ be a polynomial in n real variables, not identically 0. Let Z be the zero set of P, so $Z \subset \mathbf{R}^n$. Then $\mu_n(Z) = 0$ where μ_n is n-dimensional Lebesgue measure.

Proof. We proceed by induction on n. The theorem is clearly true when $n = 1$ since the zero set of a non-zero polynomial in one variable is finite, hence has Lebesgue measure 0. Suppose we have established the theorem for $1, 2, \ldots, n-1$.

Write $y := x_2, \ldots, x_n$ so $\mathbf{x} := (x_1, \ldots, x_n) = (x_1, y)$. Then for some N we can write P as

$$P(\mathbf{x}) = \sum_{k=0}^{N} P_k(y) x_1^k.$$

Note that $P_0, \ldots, P_N$ are polynomials in the $n-1$ variables $x_2, \ldots, x_n$, at least one of which is not identically 0. Let

$$Z_y := \{x_1 : P(x_1, y) = 0\}$$

for each $y \in \mathbf{R}^{n-1}$. (In other words, Z_y is the y-slice of Z). Finally, let

$$A := \{y \in \mathbf{R}^{n-1} : P_0(y) = P_1(y) = \cdots = P_N(y) = 0\}.$$

By induction hypothesis $\mu_{n-1}(A) = 0$. Also, for $y \notin A$, the number of elements in Z_y is less than or equal to N, so $\mu_1(Z_y) = 0$. Now $\mu_n = \mu_1 \times \mu_{n-1}$, the product measure, so by Fubini's theorem,

$$\mu_n(Z) = \int_{y \in \mathbf{R}^{n-1}} \mu_1(Z_y)\, d\mu_{n-1}(y) = \int_{y \in A} \mu_1(Z_y)\, d\mu_{n-1}(y) + \int_{y \in A^c} \mu_1(Z_y)\, d\mu_{n-1}(y) = 0.$$

This concludes the proof.