MTH 452 Selected Homework Answers

7.16 Let $y = u(x) = x^3$ so $x = w(y) = y^{1/3}$ and $Y = u(X)$. Since the support of $f(x)$ is $[0, 1]$ the support of $g(y)$, the density of Y is $[0, 8]$ and

$$g(y) = f(w(y))|w'(y)| = (y^{1/3}/2 \cdot (1/3)y^{-2/3} = (1/6)y^{1/3} \text{ for } 0 < y < 8$$

and $g(y) = 0$ otherwise.

7.18 Let's use the d.f. technique on this one. $X \sim Unif[0, 1]$ so its d.f. is

$$F(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 0 & \text{for } x \le 0 \\ 1 & \text{for } x \ge 1 \end{cases}$$

Note first that since the possible values of X are in the interval $(0, 1)$, those of $\ln X$ are in $(-\infty, 0)$ and so those of Y are in $(0, \infty)$. If $Y = -2 \ln X$ we find the d.f. of Y, $G(y)$ for $y > 0$ by
$G(y) = P(Y \le y) = P(-2 \ln X \le y) = P(\ln X \ge -y/2) = 1 - P(\ln X < -y/2) = 1 - P(X \le e^{-y/2}) = 1 - F(e^{-y/2}) = 1 - e^{-y/2}$.
Note the reversal of inequality when dividing by -2 and the preservation of the inequality when applying the (increasing) exponential function. Also note that for continuous r.v.s, we can replace $<$ with $\le$. Finally, the density of Y is $g(y) = G'(y) = (1/2)e^{-y/2}$ for $y > 0$ so $Y \sim Exp(2)$ and this is the same as $\Gamma(1, 2)$.

7.19 (DONE IN CLASS) But here's another way. $Y = u(X)$ where $u(x) = x^{-1/\alpha}$ which is decreasing with (decreasing) inverse $w(y) = y^{-\alpha}$. Since the density of X is $f(x) = 1$ for $0 < x < 1$ and observing that this implies that $y = x^{-1/\alpha} > 1$ we have the density of Y given by

$$g(y) = f(w(y))|w'(y)| = \begin{cases} 1 \cdot \alpha y^{-\alpha-1} & \text{for } y > 1 \\ 0 & \text{otherwise} \end{cases}$$