Testing the parameter for $\text{Exp}(\theta)$

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> with(Statistics) :
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Suppose $X_1,\ X_2, \ldots X_{10}$ is a sample from a $\text{Exp}(\theta)$ population and we want to test the hypothesis

H_0: $\theta = 1$

vs

H_1: $\theta > 1$.

The N-P lemma says that the most powerful test in this case
rejects H_0 when $X = X_1 + X_2 + \ldots + X_{10} > c$ for some constant c.

Since the sum of n i.i.d. $\text{Exp}(\theta)$ r.v.s is a $\Gamma(n,\theta)$ r.v. we see that under the null hypothesis H_0 we have $X \sim \Gamma(10,1)$ so if we want a size α test with $\alpha = .05$ say, we need to find c so that

$P(X > c) = .05$

when $X \sim \Gamma(10,1)$. Note

c is the *95% quantile* of the $\Gamma(10,1)$ distribution:

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> c := Quantile(Gamma(1, 10), .95);
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$$c := 15.70521642 \quad (1)$$

Note: Maple reverses the order of the parameters in the Γ density notation that we have been using!

Now we can compute the power function. This is the function

$\pi(\theta)$ = Prob of rejecting H_0 when the true parameter is θ.

$= P(X > c) = 1 - P(X < c)$ computed assuming that $X \sim \Gamma(10,\theta)$:

Note that $P(X < c)$ is the cumulative distribution function of X:

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> pi := θ→1 − CDF(Gamma(θ, 10), c);
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$$\pi := \theta \rightarrow 1 - Statistics\text{:-}CDF(\Gamma(\theta, 10), c) \quad (2)$$

Here is a plot of the power function $\pi(\theta)$

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> plot(pi(θ), θ = 1 .. 5);
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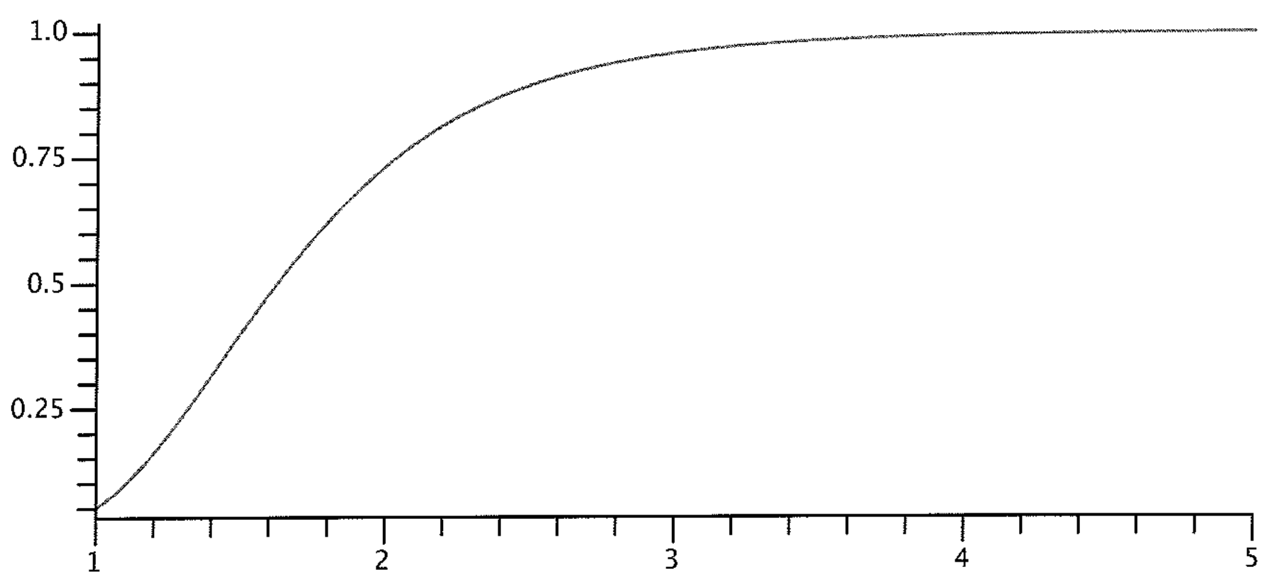


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This is all pretty sensible: If the true value of θ is actually 1 (the null hypothesis here) then each sample value has mean 1 (remember that $E(X) = \theta$ if $X \sim Exp(\theta)$), so we expect that the sum of the 10 observed sample values should be around 10 if the null hypothesis is true and we would reject the null hypothesis in favor of our alternative hypothesis that $\theta > 1$ if the observed sum is significantly larger than 10. In fact our calculation shows that if we reject when the sum is > 15.7 the probability of erroneous rejection will be .05. Suppose we did this experiment and found the observed sum to be bigger, say 19.3. We would certainly reject the null hypothesis, and if we had used the test "reject H_0 if $X > 19$" we would have rejected it for that test as well. But the size of the test "reject H_0 if $X > 19$" is $P(X > 19)$ computed assuming that $X \sim \Gamma(10,1)$ which is the value

> $1 - CDF(\Gamma(1, 10), 19.);$

$$0.0088555839 \qquad (3)$$

>

in which case the null hypothesis would have been rejected with a test of size about .009. In fact if we had chosen $c = 19.3$ exactly and found that $X = 19.3+$ we would have rejected the null hypothesis with a test of size

> $1 - CDF(\Gamma(1, 10), 19.3)$

$$0.0074734334 \qquad (4)$$

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This is the smallest size of a N-P test that would reject the null hypothesis given our actual measurement. We call it the *p-value* or the *observed significance level.*