8.16 $S^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$ (the sample variance)

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X}^2) = \sum_{i=1}^{n} X_i^2 - \underbrace{2\bar{X}\cdot\sum_{i=1}^{n} X_i}_{= 1} + \underbrace{\sum_{i=1}^{n}\bar{X}^2}_{= 2}$$

Term 1 is $= 2\bar{X}\cdot n\bar{X}$ since $n\bar{X} = \sum_{i=1}^{n} X_i$

Term 2 is $= n\bar{X}^2$ so $\sum(X_i - \bar{X})^2 = \sum_1^n X_i^2 - 2n\bar{X}^2 + n\bar{X}^2$

(since we add n $\bar{X}^2$'s) $\qquad = \sum_1^n X_i^2 - n\bar{X}^2$

Thus $S^2 = \dfrac{\sum_1^n X_i^2 - n\bar{X}^2}{n-1} = \dfrac{\sum X_i^2}{n-1} - \dfrac{n\bar{X}^2}{n-1}$.

8.18 Use mgf's. $X_i \sim \chi^2(\nu_i) = \Gamma(\frac{\nu_i}{2}, 2)$

$$M_{X_i}(t) = (1-2t)^{-\nu_i/2} \quad \text{for } i = 1, 2, \ldots, n.$$

So
$$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$
$$= (1-2t)^{-\nu_1/2} \cdots (1-2t)^{-\nu_n/2}$$
$$= (1-2t)^{-(\nu_1+\nu_2+\cdots+\nu_n)/2}$$

which is the mg.f of a $\chi^2(\nu_1+\nu_2+\cdots+\nu_n)$ r.v.

8.19 If $X_1 \sim \chi^2(\nu_1)$ and X_2, X_1 are independent with $X_1 + X_2 \sim \chi^2(\nu)$, then

$$M_{X_1}(t) \cdot M_{X_2}(t) = M_{X_1+X_2}(t) \Rightarrow$$
$$(1-2t)^{-\nu_1/2} M_{X_2}(t) = (1-2t)^{-\nu/2}$$

So
$$M_{X_2}(t) = \frac{(1-2t)^{-\nu/2}}{(1-2t)^{-\nu_1/2}} = (1-2t)^{-(\nu-\nu_1)/2}$$

$$\Rightarrow X_2 \sim \chi^2(\nu - \nu_1)$$

8.20.

Observe that $X_i - \mu = (X_i - \bar{X}) + (\bar{X} - \mu)$

so $\sum_{i=1}^{n} (X_i-\mu)^2 = \sum \left[(X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X}-\mu) + (\bar{X}-\mu)^2\right]$

$$= \sum_1^n (X_i-\bar{X})^2 + 2(\bar{X}-\mu)\sum_1^n (X_i - \bar{X}) + \sum_1^n (\bar{X}-\mu)^2$$

Note that the last term is just the sum of n $(\bar{X}-\mu)^2$'s so equals $n(\bar{X}-\mu)^2$. The middle term is 0 since $\sum_{i=1}^n (X_i - \bar{X}) = \sum_1^n X_i - \sum_1^n \bar{X}$

$$= \sum_1^n X_i - n\bar{X} = \sum_1^n X_i - \sum_1^n X_i$$

Thus $\sum (X_i-\mu)^2 = \sum (X_i - \bar{X})^2 + n(\bar{X}-\mu)^2$.

(There are many other ways to do this!)

8.46 If $X_1, \ldots, X_n \sim \text{Unif}[0,1]$ the common d.f. is $F(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \le 0 \\ 1, & x > 1 \end{cases}$

and common density is $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0 & \text{else} \end{cases}$

so the density of Y_1 is $n(1-F(y))^{n-1} f(y) = \begin{cases} n(1-y)^{n-1}, & 0 < y < 1 \\ 0 & \text{else} \end{cases}$.

Thus $EY_1 = \int_0^1 y \cdot n(1-y)^{n-1}\,dy = \frac{1}{n+1}$

$$EY_1^2 = \int_0^1 y^2 n(1-y)^{n-1}\,dy = \frac{2}{(n+1)(n+2)}$$

$$Var(Y_1) = E(Y_1^2) - (EY_1)^2 = \frac{n}{(n+1)^2(n+2)}$$

The easy way to do these integrals is to substitute $u = 1-y$, Then $du = -dy$ and $y = 1-u$ so

e.g. $$\int_0^1 y^2(1-y)^{n-1}\,dy = -\int_1^0 (1-u)^2 u^{n-1}\,du$$

$$= \int_0^1 (1-2u+u^2)\,u^{n-1}\,du$$

$$= \int_0^1 u^{n-1}\,du - 2\int_0^1 u^n\,du + \int_0^1 u^{n+1}\,du$$

$$= \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$

So $EY_1^2 = n\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = \frac{2}{(n+1)(n+2)}$

8.67 If $X_1, \dots, X_{200} \sim \text{Unif}[24, 48]$

then $\mu = EX_1 = \frac{48+24}{2} = 36$ and $\sigma^2 = \text{Var}(X_1) = \frac{24^2}{12} = 48$

So $\sigma = \sqrt{48}$ and

$$P(\bar{X} < 35) = P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{200}} < \frac{35-36}{\sqrt{48}/\sqrt{200}}\right)$$

by CLT $\simeq P\left(Z < \frac{-1}{\sqrt{\frac{48}{200}}}\right) \doteq P(Z < -2.041)$

$= .0206$

$N(0,1)$