MTH 452 2/21/08 Name (Print):________________________

1. Suppose $X_1, \dots, X_n$ is a sample from a population with density

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $\hat{\theta}$ is an unbiased estimate of θ based on this sample, show that the variance of $\hat{\theta}$ is greater than or equal to θ^2/n

You may use without proof the two integrals

$$\int_0^1 \ln(x)x^{\theta-1}\,dx = -1/\theta^2, \qquad \int_0^1 (\ln(x))^2 x^{\theta-1}\,dx = 2/\theta^3.$$

Solution: $\ln f(x;\theta) = \ln\theta + (\theta-1)\ln x$ so

$$\frac{\partial \ln f(x;\theta)}{\partial\theta} = \frac{1}{\theta} + \ln x$$

The given integrals show that $E(\ln X) = \int_0^1 \ln x \cdot \theta x^{\theta-1}\,dx = -1/\theta$ and $E[(\ln X)^2] = \int_0^1 (\ln x)^2 \cdot \theta x^{\theta-1}\,dx = 2/\theta^2$

so

$$E[(\frac{\partial \ln f(X;\theta)}{\partial\theta})^2] = E[(\frac{1}{\theta} + \ln X)^2]$$

$$= E[\frac{1}{\theta^2} + 2\frac{1}{\theta}\ln X + (\ln X)^2] = \frac{1}{\theta^2} - 2\frac{1}{\theta^2} + \frac{2}{\theta^2} = \frac{1}{\theta^2}$$

so by the Cramer-Rao inequality

$$Var(\hat{\theta}) \geq \frac{1}{n/\theta^2}.$$

2. Suppose X_1, X_2 are i.i.d. with mean μ. Show that for any real number α with $0 < \alpha < 1$

$$\alpha X_1 + (1-\alpha)X_2 \tag{1}$$

is an unbiased estimator of μ and that for all estimators of this form the one with minimum variance is $\frac{1}{2}X_1 + \frac{1}{2}X_2$, i.e. when $\alpha = \frac{1}{2}$.

Solution: $E(\alpha X_1+(1-\alpha)X_2) = \alpha E(X_1)+(1-\alpha)E(X_2) = \alpha\mu+(1-\alpha)\mu = \mu$ which shows that (1) is unbiased.
Since X_1 and X_2 have the same variance σ^2, the variance of (1) is

$$\alpha^2 Var(X_1) + (1-\alpha)^2 Var(X_2) = \alpha^2\sigma^2 + (1-\alpha)^2\sigma^2 = (2\alpha^2 - 2\alpha + 1)\sigma^2.$$

But $2\alpha^2 - 2\alpha + 1$ is minimized when its derivative is 0, that is, when $4\alpha - 2 = 0$, or $\alpha = 1/2$.