MTH 452 2/11/08 Name (Print): Key

1. Suppose the r.v. X has density

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and so has d.f. $F(x) = 1 - e^{-x}$ for $x > 0$. Find the density of the r.v. $Y = 1/X$. (Note: Remember that $1/x$ is a *decreasing* function of x.)

$Y > 0$. For $y > 0$, $P(Y \le y) = P(\frac{1}{X} \le y) = P(\frac{1}{y} \le X)$

$= 1 - P(X \le \frac{1}{y}) = 1 - F(\frac{1}{y}) = 1 - (1 - e^{-\frac{1}{y}})$

$= e^{-\frac{1}{y}} = G(y)$, the d.f. of Y.

Then the density of Y, for $y > 0$ is

$g(y) = G'(y) = \frac{1}{y^2} e^{-1/y}$. Ans. $g(y) = \begin{cases} \frac{1}{y^2} e^{-\frac{1}{y}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

2. Suppose $X_1, X_2 \overset{\text{i.i.d.}}{\sim} \Gamma(\frac{1}{2}, 1)$. Let $Y = X_1 + X_2$.
Find $P(Y \le 1)$. Hint: Use m.g.f.'s to identify the density of Y.

The m.g.f. of X_1 and X_2 is $(1-t)^{-1/2}$ so the mgf. of Y is $(1-t)^{-1/2} \cdot (1-t)^{-1/2} = (1-t)^{-1}$ which is the mgf of a $Exp(1)$ r.v. Hence the density of Y is $f(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

Finally, $P(Y \le 1) = \int_0^1 f(y)dy = \int_0^1 e^{-y} dy = -e^{-y}\Big|_0^1$

$= -e^{-1} - (-1)$

$= 1 - \frac{1}{e}$.