Applications of the Definite Integral

Areas. Volumes. Arc Length. Total Mass

The definite integral is used to calculating areas, volumes, arc length, the total mass given its density, etc. As you know, very often the main difficulty in such applications is setting up an appropriate integral. The *plot* and *solve* functions can help determine limits of integration and the *int* can find integrals in algebraic or numerical form. This can be especially useful when the integrals are not readily handled by the fundamental theorem of calculus.

**Example 1.** Consider the region R bounded by the graphs of the functions \( f(x) = e^{-x^2} \cos(x) \) and \( g(x) = x^2 \).

(a) Find the area of the region R.
(b) Find the perimeter of R.
(c) Find the volume of the solid obtained by revolving the region R about the x axis.

The units on each of the axes are centimeters.

We will use the following Maple commands:

\[
\begin{align*}
\text{int} & (f(x), x = a \ldots b); & \text{Integrates } f(x) \text{ from } x = a \text{ to } x = b \\
\text{Int} & (f(x), x = a \ldots b); & \text{Forms the integral but does not evaluate it} \\
D(f)(x) & ; & \text{Differentiates } f(x) \\
evalf( & \exp(x)); & \text{Evaluates } \exp(x), \text{ giving a real number} \\
solve & (f(x) = 0, x); & \text{Solves the equation } f(x) = 0 \text{ for } x \\
\text{plot} & ([f(x), g(x)], x = a \ldots b); & \text{Plots graphs of } f(x), g(x) \text{ on same axes}
\end{align*}
\]

First define the functions \( f \) and \( g \) in Maple. (Note the use of the semicolon to separate commands here.)

\[
\begin{align*}
f & := x \mapsto \exp(-x^2) \cos(x) \\
g & := x \mapsto x^2
\end{align*}
\]

Next plot the graphs of \( f \) and \( g \), using red for \( f \) and blue for \( g \), for \( x \) in the interval from -5 to 5. We choose these numbers because we need to start somewhere!

\[
\text{plot}([f(x), g(x)], x = -5 \ldots 5, \text{color} = [\text{red, blue}]);
\]
We now see where the region R is, assuming that we are interested in the region with the red graph above
the blue one. The picture will be much clearer in a smaller range for x. We now create a better plot, and
even give it a title:

```
plot([f(x), g(x)], x=-1..1, color=[red, blue], title="REGION R");
```

Obviously, to answer any of the questions (a)-(c), we have to find the points of intersection of the two
graphs. The equations involved cannot be solved algebraically, or by using the "solve" command that
attempts to find exact solutions. We shall use the "fsolve" command that gives numerical answers, and
allows us to specify the range in which we want a solution. Let's label the points of intersection by "a" and
"b", respectively, with "a" between -1 and 0 and "b" between 0 and 1:

```
a := fsolve(f(x) = g(x), x,-1..0); b := fsolve(f(x) = g(x), x, 0..1)
```

(2)
Not surprisingly, \( a = -b \). Observe that \( f(x) \) and \( g(x) \) are both even. Hence, region \( R \) is symmetric with respect to the \( y \) axis. With the points of intersection found, we are prepared to set up the definite integral that represents the area of the region. We label the area "\( A \)". Note that the graph of \( f \) is above the graph of \( g \) so we want to integrate \( f(x) - g(x) \).

\[
A := \text{Int}(f(x) - g(x), x = a..b);
\]

\[
\int_{-0.6912304114}^{0.6912304114} \left( e^{-x^2} \cos(x) - x^2 \right) \, dx \tag{3}
\]

Notice that we used a capital letter "\( I \)" in the integral command "\( \text{Int.} \)". This resulted in Maple's displaying our integral in standard printed form.

This integral cannot be determined using the Fundamental Theorem, that is, by finding an antiderivative of \( e^{-x^2} \cos(x) - x^2 \). Hence, we shall use the \( \text{evalf} \) command that, whenever applied, provides a numerical approximation.

\[
\text{evalf}(A);
\]

\[
0.8889316488 \tag{4}
\]

The area of our region is approximately \( 0.889 \, \text{cm}^2 \). We could have just used "\( \text{int} \)" with a lower case "\( i \)" in (3) to get the same result immediately.

To find the perimeter \( P \) of the region we have to add the arc length of the two pieces. We use the familiar formula for the arc length to set up the appropriate integrals. Recall that "\( D \)" stands for the derivative function. We will call the length of the lower graph (\( g \) here), \( P_1 \), and the length of the upper graph, \( f \), \( P_2 \). As above we use the "\( \text{Int} \)" command to look at the integrals in standard printed form before finding numerical values with \( \text{evalf} \).

\[
P_1 := \text{Int}(\sqrt{1 + D(f)(x)^2}, x = a..b);
\]

\[
P_2 := \text{Int}(\sqrt{1 + D(g)(x)^2}, x = a..b);
\]

\[
\int_{-0.6912304114}^{0.6912304114} \sqrt{1 + \left( -2 \cdot x \cdot e^{-x^2} \cos(x) - e^{-x^2} \sin(x) \right)^2} \, dx
\]

\[
\int_{-0.6912304114}^{0.6912304114} \sqrt{1 + 4 \cdot x^2} \, dx \tag{5}
\]

\[
\text{evalf}(P_1 + P_2)
\]

\[
3.520790977 \tag{6}
\]

The perimeter is approximately equal to 3.521 cm.

To find the volume of revolution, we slice the solid perpendicularly to the \( x \) axis. The approximate volume of each "washer" slab is \( \pi f(x)^2 \Delta x - \pi g(x)^2 \Delta x \). The corresponding volume of the solid is then given by the integral

\[
V := \text{int} \left( \pi \left( f(x)^2 - g(x)^2 \right), x = a..b \right)\]
\[ \int_a^b \pi \left( (e^{-x^2})^2 \cos(x)^2 - x^4 \right) \, dx \]

\[ \text{evalf}(V) ; \]

\[ \int_a^b \pi (f(x)^2 - g(x)^2) \, dx \] \hspace{1cm} (8)

The volume is approximately 2.712 cm\(^3\).

**Parametric Curves**

**Example 4.** Satellites in orbit around a central mass have elliptical orbits. For instance, the orbits of the planets around the sun are elliptical (though nearly circular) while the orbits of artificial satellites around the earth are often very elliptical. This example concerns a motion along an elliptical path. As you know, motion is best described using parametric representations of curves.

Consider a point moving in the plane so that its x and y coordinates are functions of time t given as follows.

\[ X := t \rightarrow 2 \cos(t) ; \quad Y := t \rightarrow 3 \sin(t) ; \]

\[ t \rightarrow 2 \cos(t) \]

\[ t \rightarrow 3 \sin(t) \] \hspace{1cm} (9)

Note that we have used capital X and Y because using the lower case letters would make them unavailable for use as variables in Maple. The point (X,Y) will describe an ellipse as t goes from 0 to \(2 \pi\). We plot the elliptical path below. \textit{Note the syntax appropriate for parametric plots, especially the location of square brackets.} In order to see clearly the elliptical shape of the path, we use the "\texttt{scaling=constrained}" option, which makes the scales on the vertical and horizontal axes the same. Execute the following command.

\[ \text{plot(} [X(t), Y(t), t = 0..2 \pi], \text{scaling = constrained}) ; \]
Suppose that your position at time $t$ is given by the above parametric equations.

(a) What is your position and speed at $t=3.5$?
(b) How far along the ellipse do you move between time $t=0$ and $t=3.5$?
(c) Given that you started at $(2,0)$ when $t=0$, at what time will you have traveled exactly 4 units of length along the ellipse?

To answer (a), observe that the position at 3.5 is $(X(3.5), Y(3.5))$, and by the familiar formula, the speed at time $t$ is

$$v(t) = \sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2}.$$ 

Hence, we obtain

$$X(3.5), Y(3.5); \quad -1.872913375, -1.052349683 \quad (10)$$

Our position at $t=3.5$ is, approximately, (-1.873, -1.052).

We set up a function that gives the speed $v$, as a function of $t$. We use the "D" symbol for derivative so that, for example, $X'(t) = D(X)(t)$.

$$v := t \mapsto \sqrt{D(X)(t)^2 + D(Y)(t)^2};$$

$$t \mapsto \sqrt{D(X)(t)^2 + D(Y)(t)^2} \quad (11)$$

The speed at $t=3.5$ is then

$$v(3.5); \quad 2.895644252 \quad (12)$$
To answer (b) we need the formula that says that the arc length of a parametric curve is the integral of the speed. We use it below to calculate the distance traveled between \( t=0 \) and \( t=3.5 \). This time we won't display the integral, but just calculate its value using \( \text{int} \).

\[
\text{int}(v(t), t = 0..3.5); \quad 8.995345610
\]  (13)

We have traveled almost 9 units of length. To answer (c) we have to set up an equation and solve it numerically. The length traveled up to time \( T \) is given by the integral of \( v(t) \) from 0 to \( T \). We define this as a function, called \( L \), of \( T \). We use \( \text{Int} \) instead of \( \text{int} \) here because using \( \text{Int} \) tells Maple not to actually calculate the integral until needed.

\[
L := T \rightarrow \text{Int}(v(t), t = 0..T);
\]

We want this integral to equal 4, so we have to solve numerically for \( T \) the equation \( L(T)=4 \). Since \( L(T) \) is strictly increasing there is only one solution to this equation and we do not have to worry about specifying the range for a solution.

\[
\text{fsolve}(L(T) = 4, T); \quad 1.587615387
\]  (15)

At \( t \) equal approximately to 1.588 units of time, we have traveled 4 units of distance.
Homework Problems

Instructions: Create a new worksheet (or use the template worksheet provided). Be sure that your name appears *printed within Maple* at the top of the first page. Then solve the following problems using Maple for all calculations and graphs, and provide clear text to explain what you have done.

**Problem 1.** Consider the region G bounded by the graphs of \( y = 2 \cos(x) \) and \( y = \frac{x^4}{2} \). Units on both axes are inches.
(a) Plot the region.
(b) Express the perimeter of G as a definite integrals and find its numerical value.
(c) Express the area of G as definite integrals (or one integral) and find the numerical value of the area of G.
(d) Express the volume of the solid obtained by revolving region G about the x axis as an integral and compute its value.
(e) Suppose that region G defined has varying mass density given by \( m(x) = 5(x^2 + 1) \) grams per inch square. Express its total mass in terms of definite integrals and find the numerical value of the total mass.

**Problem 2.** A particle is traveling along the path in the xy plane given by the parametric equations
\[ x = 3 \sin(2t), \quad y = \cos(t), \]
where \( t \) is time.
(a) Plot the path for \( t \) in the interval \([0,2]\), then for \( t \) in the interval \([0,3]\), then \([0,5]\) and \([0,7]\). Describe the motion of the particle.
(b) What is the particle's position and speed at \( t=3.5 \)?
(c) What distance does the particle travel before it returns to its starting point at \((0,1)\)?
(d) At what time will the particle have travelled 5 units from its starting point?

**Problem 3.** Suppose a region R is bounded by the x-axis, the y-axis, and the graph of \( y = 10 - xe^{-x} \).
(a) Plot this region.
b) Find the volume of the solid of revolution you obtain if you rotate this region around the x-axis.