MTH 142 Solution Practice for Exam 2 Updated 10/19/2004

1. (a) $\Delta x = 4/3$, hence $\text{MID}(3) = \left(\frac{1}{1+2/3} + \frac{1}{1+6/3} + \frac{1}{1+10/3}\right)\left(\frac{4}{3}\right) = 1.5512$

   (b) $\text{LEFT} = (-4 - 2.25 - 1 - 0.25)0.5 = -3.75$ and $\text{RIGHT} = (-2.25 - 1 - 0.25 - 0)0.5 = -1.75$. Hence TRAP $= -2.75$.

2. (a) $\text{MID} = (1.492 + 2.48 + 2.92 + 2.98)0.5 = 4.936$.
   (b) $\text{TRAP} = (3.915 + 5.345)/2 = 4.63 \Rightarrow \text{SIMP} = (2 \cdot 4.936 + 4.63)/3 = 4.834$.

3. (a) RIGHT, MID, TRAP, LEFT. Reason: since the function is decreasing, we have RIGHT $<$ LEFT. Also, TRAP is the average of RIGHT and LEFT. In addition, the function is concave up, which implies MID $<$ TRAP.

   (b) Errors: exact - right(5) = .0032, exact - mid(5) = .0012, exact - trap(5)= -.0005, exact - left (5) = -.0043.

   (c) LEFT and RIGHT: the first 3 decimals will be correct since the error improves by 1 decimal place. MID: the first 4 decimals will be correct since the error is improved by 2 decimal places. TRAP: the first 5 decimals will be correct since the error is improved by 2 decimal places.

4. (a) Improper. $\displaystyle\int_1^\infty \frac{3}{\sqrt{2+x}} = \lim_{b\to\infty}\int_1^b 3(2+x)^{-1/2}\,dx = \lim_{b\to\infty} 6(2+x)^{1/2}\Big|_1^b =$
   $\displaystyle= \lim_{b\to\infty} 6\sqrt{2+b} - 6\sqrt{3} = \infty$. Hence the integral diverges.

   (b) Improper. $\displaystyle\int_{-1}^5 \frac{2}{2x+2}\,dx = \lim_{c\to -1^+}\int_c^5 \frac{1}{x+1}\,dx = \lim_{c\to -1^+} \ln|x+1|\big|_c^5$
   $\displaystyle= \lim_{c\to -1^+} \ln 6 - \ln|c+1| = \infty$. Hence the integral diverges.

   (c) Improper. $\displaystyle\int_0^5 \frac{2}{t^2+3t}\,dt = \lim_{c\to 0^+}\int_c^5 \frac{2}{t(t+3)}\,dt \lim_{c\to 0^+} = \int_c^5 \frac{2/3}{t} - \frac{2/3}{t+3}\,dt =$
   $\displaystyle= \lim_{c\to 0^+} \frac{2}{3}\ln|t| - \frac{2}{3}\ln|t+3|\Big|_c^5 = \lim_{c\to 0^+} \frac{2}{3}\ln\left|\frac{t}{t+3}\right|\Big|_c^5 = \lim_{c\to 0^+} \frac{2}{3}\ln\left|\frac{5}{8}\right| - \frac{2}{3}\ln\left|\frac{c}{c+3}\right| = \infty$ Hence the integral diverges.

   (d) Improper. $\displaystyle\int_{-\infty}^\infty e^{3t}\,dt = \int_{-\infty}^0 e^{3t}\,dt + \int_0^\infty e^{3t}\,dt = \text{(I)} + \text{(II)}$.
   We now analyze (I) and (II) separately:
   $\displaystyle\text{(I)} = \lim_{c\to -\infty}\int_c^0 e^{3t}\,dt = \lim_{c\to -\infty} \frac{1}{3}e^{3t}\Big|_c^0 = \lim_{c\to -\infty} \frac{1}{3} - \frac{1}{3}e^c = \frac{1}{3} - 0$. Hence (I) converges.
   $\displaystyle\text{(II)} = \lim_{b\to -\infty}\int_0^b e^{3t}\,dt = \lim_{b\to\infty} \frac{1}{3}e^{3t}\Big|_0^b = \lim_{b\to\infty} \frac{1}{3}e^b - \frac{1}{3} = \infty$. Hence (II) diverges.
   We conclude that $\displaystyle\int_{-\infty}^\infty e^{3t}\,dt$ also diverges.

5. (a) "Behaves-like" analysis: $\displaystyle\frac{x}{\sqrt{1+x^6}} \approx \frac{x}{\sqrt{x^6}} = \frac{1}{x^2}$ when x is large (p=2). Hence we suspect convergence. We now compare the integrand with a larger function whose integral converges. We note that $\sqrt{1+x^6} \geq \sqrt{x^6} = x^3$ for $x \geq 1$, which implies that the following inequality is valid: $0 \leq \frac{x}{\sqrt{1+x^6}} \leq \frac{x}{\sqrt{x^6}} = \frac{1}{x^2}$, for $1 \leq x < \infty$. We conclude from the comparison test that $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ converges.

(b) "Behaves-like" analysis: $\dfrac{t^2+1}{t^2-1} \approx \dfrac{t^2}{t^2} = 1$ for large t (p=0). Hence we suspect divergence. We now compare the integrand with a smaller function whose integral diverges; For this we note that $t^2 < t^2+1$ and that $t^2 > t^2-1$, which imply that the following inequality is valid: $1 = \dfrac{t^2}{t^2} \le \dfrac{t^2+1}{t^2-1}$ for $2 \le t < \infty$. We conclude from the comparison test that $\int_2^\infty \frac{t^2+1}{t^2-1}\,dt$ diverges.

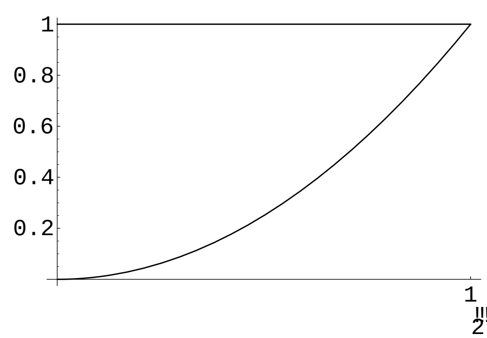
6. By taking sections perpendicular to the axis of rotation, we get "washers". At the tickmark x_j the washer has inner radius $r_j = 2x_j^2$, outer radius $R_j = 1$, and thickness Δx. The sum that approximates the volume is

$$V \approx \sum_{j=0}^{n} (\pi R_j^2 - \pi r_j^2)\Delta x = \sum_{j=0}^{n} (\pi 1^2 - \pi(2x_j^2)^2)\Delta x$$

The exact volume is obtained by taking limit as $\Delta x \to 0$. We have,

$$Vol(S) = \int_0^{\sqrt{2}/2} (\pi - \pi 4x^4)dx = \frac{2\sqrt{2}\pi}{5} \approx 1.777153$$

7.



By taking sections perpendicular to the axis of rotation, we get "disks". At the tickmark y_j the radius is $r_j = x_j = \sqrt{y_j/2}$ and the thickness is Δy.

The sum that approximates the volume is

$$\sum_{j=0}^{n} \pi R_j^2 \Delta y = \sum_{j=0}^{n} \pi(\sqrt{y_j/2})^2 \Delta y = \sum_{j=0}^{n} \pi y_j/2 \Delta y$$

The volume is obtained by taking limit as $\Delta y \to 0$. We have,

$$Vol(S) = \int_0^1 \frac{\pi}{2} y dy = \frac{\pi}{4}$$

8. By taking sections perpendicular to the axis of rotation, we get "washers". At the tickmark y_j the washer has inner radius $r_j = 1$, outer radius $R_j = 1 + \sqrt{y_j/2}$, and thickness Δy. The sum that approximates the volume is

$$V \approx \sum_{j=0}^{n} (\pi R_j^2 - \pi r_j^2)\Delta y = \sum_{j=0}^{n} (\pi(1+\sqrt{y_j/2})^2 - \pi(1)^2)\Delta y$$

The volume is obtained by taking limit as $\Delta x \to 0$. We have,

$$Vol(S) = \int_0^1 (\pi(1+\sqrt{y/2})^2 - \pi)dy = \pi(\frac{1}{4} + \frac{2\sqrt{2}}{3}) \approx 3.74732$$

9. a)

$$\sum_{j=0}^{n} \frac{0.004}{1+r_j^2} 2\pi r \Delta r$$

b)

$$\int_0^{7000} \frac{(0.004)2\pi r}{1+r^2} \Delta r$$

10. a)

$$\sum_{j=0}^{n} (2+0.015x)\Delta x$$

b)

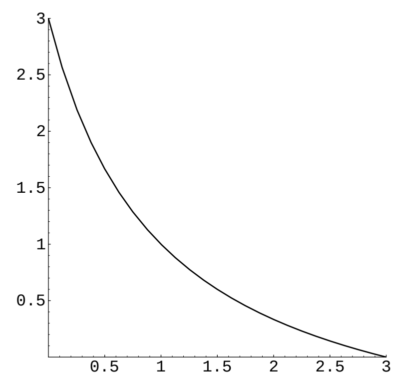
$$\int_0^1 (2+0.015x)dx = 2.0075$$

c)

$$\overline{x} = \frac{\int_0^1 x\,(2+0.015x)dx}{\int_0^1 (2+0.015x)dx} = \frac{1.0050}{2.0075} = 0.50062266$$

11.

A plot of $y = (3-x)/(1+x)$ produced with a graphing calculator shows that the plate has the shape shown in the figure. It is clear that the curve meets the X and Y axes at $x = 3$ and $y = 3$ respectively. Since the plate has constant density, the formulas in page 365 of the text apply.



The total mass of the plate is Mass $= \int_0^3 0.15\, \frac{3-x}{1+x}\, dx \approx 0.381777$. The center of mass $(\overline{x}, \overline{y})$ is given by

$$\overline{x} = \frac{\int_0^3 x\, 0.15\, \frac{3-x}{1+x}\, dx}{Mass} = \frac{0.293223}{0.381777} \approx 0.768062$$

By symmetry we have that $\overline{y} = \overline{x} = 0.768062$. Hence $(\overline{x}, \overline{y}) = (0.768062, 0.768062)$.
Note: to obtain $\overline{y}$ with a calculation, it may be done as follows:
Solve for x in $y = (3-x)/(1+x)$ to obtain $x = (3-y)/(1+y)$. Then,

$$\overline{y} = \frac{\int_0^3 y\, 0.15\, \frac{3-y}{1+y}\, dy}{Mass} = \frac{0.293223}{0.381777} \approx 0.768062$$

12. Slice the drum with horizontal, circular sections. Each section corresponds to a tickmark y_j on the vertical axis. The number of bacteria in a section at tickmark y_j is

$$Number(Section_j) \approx \text{density} \cdot \text{volume} = (3.5 - 0.05y_j)\ \pi(24)^2 \Delta y$$

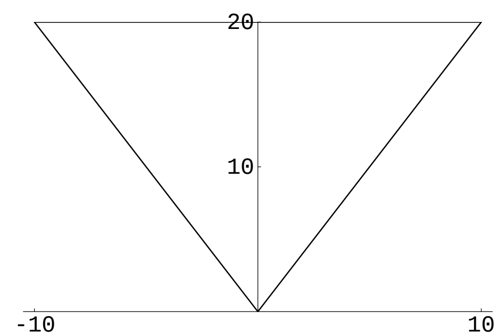
The total number of bacteria is approximated by the Riemann Sum

$$Number(Container) = \sum_{j=1}^{n} (3.5 - 0.05y_j)\ \pi(24)^2 \Delta y$$

The exact number is obtained by passing to the limit as $\Delta y \to 0$. It is,

$$\int_0^{36} (3.5 - 0.05y)\ \pi(24)^2 dy = 169375 \quad \text{million bacteria}$$

13.



A cross-section of the cone (shown in the figure) is bounded by the lines $y = \pm 2x$ and $y = 20$. Introduce tick marks in the y-axis.

The slab S_j at height y_j is a disk with radius $R_j = x_j = y_j/2$ and thickness Δy, so its volume is $\pi(y_j/2)^2\Delta y$, and its weight is $62.4\,\pi(y_j/2)^2\Delta y$. The work involved in raising the slab a distance of $(20 - y_j)$ to the top of the cone is

$$w_j = (20 - y_j)\,62.4\,\pi(y_j/2)^2\Delta y$$

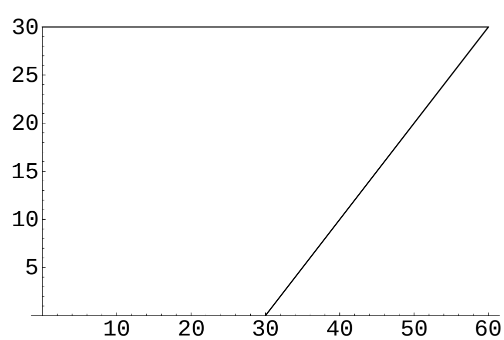
The total work is approximated by

$$W \approx \sum_{j=0}^{n} (20 - y_j)\,62.4\,\pi(y_j/2)^2\Delta y$$

The exact work is given by

$$\int_0^{20} (20 - y)\,62.4\,\pi(y/2)^2\Delta y = 653451.2720$$

14. A sketch of the dam is shown in the figure below.



Note that the equation of the right hand, non-horizontal side is $y = x - 30$. Introduce tick marks $y_0, y_1, \ldots, y_n$, in the y axis. At height y_j, the slab has area $\approx (y_j + 30)\Delta y$, and the pressure at this height is $62.4(30 - y_j)$. Therefore the force on the slab is

$$F_j = 62.4(30 - y_j)(y_j + 30)\Delta y$$

The total force is approximated by

$$F \approx \sum_{j=0}^{n} 62.4(y_j + 30)(30 - y_j)\Delta y$$

The exact value of the total force is obtained by taking the limit as $\Delta y \to 0$:

$$F = \int_0^{30} 62.4(y_j + 30)(30 - y_j)dy = 1,123,200$$

15. a) $\displaystyle\int_{1.5}^{1.7} \frac{2}{x^2}dx = 0.1569$

    b) $\displaystyle\int_{1.5}^{2} \frac{2}{x^2}dx = \frac{1}{3}$

16. a) $\displaystyle\int_1^T \frac{2}{x^2}dx = 0.5 \Longrightarrow \frac{-2}{x}|_1^T = 0.5 \Longrightarrow \frac{-2}{T} + 2 = 0.5 \Longrightarrow T = \frac{4}{3}$

    $\overline{x} = \int_1^2 x\, \frac{2}{x^2}dx = \int_1^2 \frac{2}{x}dx = 2\ln(2)$

    b) $\displaystyle\int_0^T 2e^{-2x}\,dx = 0.5 \Longrightarrow -e^{-2x}\Big|_0^T = 0.5 \Longrightarrow -e^{-2T} + 1 = 0.5 \Longrightarrow T = \frac{\ln 0.5}{-2} \approx 0.346574$

    $\displaystyle\overline{x} = \int_0^\infty x\,2e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b 2\,x\,e^{-2x}\,dx = \lim_{b\to\infty} -\frac{x}{e^{2\,x}} + \frac{-1}{2\,e^{2\,x}}\Big|_0^b =$

    $\displaystyle= \lim_{b\to\infty}(-\frac{b}{e^{2\,b}} + \frac{-1}{2\,e^{2\,b}}) - (0 - \frac{1}{2}) = \frac{1}{2}$

17. a) The cumulative distribution function is an antiderivative of the density, so $P = \int \frac{2}{x^2}dx = -\frac{2}{x}+c$. We now find c. We also know that $P = 0$ when $x = 1$ (first value of x). Substituting we find $c = 2$, so $P(x) = -\frac{2}{x} + 2$.

    Another way to solve it: $P(x) = \int_1^x \frac{2}{t^2}dt = -\frac{2}{t}\Big|_1^x = -\frac{2}{x} + 2$.

    b) $P = \int 2e^{-2x}\,dx = -e^{-2x} + C$. We also know that $P = 0$ when $x = 0$. Substituting into P we have $0 = -1 + C$, that is, $C = 1$. We conclude that $P = -e^{-2x} + 1$.

    Another way to solve it: $P(x) = \int_0^x 2e^{-2t}dt = -e^{-2t}|_0^x = -e^{-2x} + 1$.

18. The increasing function is the Cumulative Distribution Function, which we know has range from 0 to 1. This gives the vertical range, so the tick marks on the y axis are at $\{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$. Also, we know that the region under the density function (approx. 2.5 rectangles) has area 1. Then each rectangle has area $1/2.5 = 0.4$. But we also know the height of the rectangle is 0.2. Then the base of the rectangle is approximately 0.4/0.2 = 2. Therefore the tick marks on the x-axis are at $\{0, 2, 4, 6, 8, 10\}$.