MTH 142 Practice Problems for Exam 3 -Spring 2004

Last changed: April 1, 2004, 9:00 a.m

Sections 9.1, 9.2, 9.3, 9.4, 10.1, 10.2, 10.3, 10.4, 10.5

1. Obtain the first three nonzero terms of the Taylor series of $f(x) = \sqrt{x}$ about 4.

2. Obtain $P_3(x) =$ the Taylor polynomial of order 3 of $\tan x$ about $a = \pi/4$

3. Calculate the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} n3^n(x-2)^n$.

4. Given that $R = 2$ is the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}(x+1)^n$, find the interval of convergence (include endpoint analysis).

5. Calculate the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{n+3}x^n$.

6. Given that $R = 1$ is the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}x^n$, find the interval of convergence (include endpoint analysis).

7. A certain amount of fresh water shrimp is placed in a tank together with 2 lbs. of food, at 12:00 noon on January 1st, and 2 lbs. of food are added to the tank at noontime every day after Jan 1. After every 24 hours, 15% of the food either decomposes or is eaten.

   a) How much food is in the tank right after 12:00 noon on January 20 th? Give details, and explain how you arrived at your answer.

   b) In the long run, how much food is in the tank right after noontime? (Section 9.1)

8. The degree 2 Taylor polynomial approximation of $f(x) = \ln(1+x)$ for x near 0 is $P_2(x) = x - \frac{1}{2}x^2$. We wish to approximate $\ln(1.5)$ by $P_2(x)$.

   a) What should x be? What is the error? (use your calculator)

   b) Find a bound for the error using the formula studied in class.

9. a) Use the method of substitution to find the first 4 nonzero terms of the Taylor series of the function $f(x) = \frac{1}{\sqrt{1+x^2}}$ about $x = 0$.

   b) Use series to answer the following question: For values of x that are close to 0, which function is larger, $\cos(x)$ or $\frac{1}{\sqrt{1+x^2}}$?

10. Calculate the order 3 Fourier polynomial of the 2π periodic function given on $[-\pi, \pi)$ by

$$f(t) = \begin{cases} -0.5 & \text{if } -\pi \le t < 0 \\ 1 & \text{if } 0 \le t < \pi \end{cases}$$

11. Calculate the order 3 Fourier polynomial of the function given on $[-\pi, \pi)$ by $f(t) = 3t$.

Use the suggested method to determine if the series converges

12. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{1+n^3}$ (a) Comparison Test. (b) Integral Test.

13. $\displaystyle\sum_{n=1}^{\infty} 1 + \frac{1}{n}$ (a) comparison test. (b) integral test

14. $\displaystyle\sum_{n=3}^{\infty} \frac{(-2)^{n+1}}{\pi^n}$ (a) Geometric series. (b) alternating series test. (c) ratio test.

SOLUTION MTH142 Practice Problems for Exam 2

1.
$$T(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 + \cdots$$

2.
$$P_3(x) = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3$$

3. Using the ratio test we have
$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(n+1)3^{n+1}(x-2)^{n+1}}{n3^n(x-2)^n} \right| = \lim_{n\to\infty} \frac{(n+1)3|x-2|}{n} = 3|x-2|$$
Now $L < 1$ when $3|x-2| < 1$, that is, $|x-2| < \frac{1}{3}$. Then the radius of convergence is $\frac{1}{3}$.

4. Given that $R = 2$ is the radius of convergence, we note that the "base point" is $a = -1$, hence the interval of convergence goes from -3 to 1. We now analyze convergence at the endpoints. (a) Substituting $x = -3$ into $\sum_{n=0}^{\infty} \frac{1}{n^2 2^n}(x+1)^n$, we have the series $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ The absolute convergence test guarantees the convergence of the series provided the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, which does converge (p-series, with $p = 2$). Hence $x = -3$ belongs to the interval of convergence. (b) Now substitute $x = 1$ into $\sum_{n=0}^{\infty} \frac{1}{n^2 2^n}(x+1)^n$, to obtain the series $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}(2)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges ($p = 2$ again). Conclusion: the interval of convergence is $-3 \le x \le 1$ (or, in interval notation, $[-3, 1]$.)

5. Using the ratio test we have
$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{\frac{n+2}{n+4}x^{n+1}}{\frac{n+1}{n+3}x^n} \right| = \lim_{n\to\infty} \frac{(n+2)(n+3)|x|}{(n+1)(n+4)} = \lim_{n\to\infty} \frac{(n^2+5n+6)|x|}{n^2+5n+4} = 1|x|$$
Now $L < 1$ when $|x| < 1$. Then the radius of convergence is 1.

6. Given that $R = 1$ is the radius of convergence, we note that the "base point" is $a = 0$, hence the interval of convergence goes from -1 to 1. We now analyze convergence at the endpoints. Substituting $x = -1$ into $\sum_{n=0}^{\infty} \frac{n}{n^2+1}x^n$, we obtain the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}(-1)^n$ which is an alternating series. Since $\frac{1}{1^2+1} \ge \frac{2}{2^2+1} \ge \frac{3}{3^2+1} \ge \cdots$ and also $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, the series converges. Therefore $x = -1$ is part of the interval of convergence. To test the other endpoint, substitute $x = 1$ into the power series to obtain $\sum_{n=1}^{\infty} \frac{n}{n^2+1}(1)^n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ This is a divergent series, since $f(x) = \frac{x}{x^2+1}$ is decreasing and positive, we may apply the integral test for series: we get: $\int_1^{\infty} \frac{x}{x^2+1}dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}dx = \lim_{b\to\infty} \left| \frac{1}{2}\ln|x^2+1| \right|_1^b = \lim_{b\to\infty} \frac{1}{2}\ln|b^2+1| - \frac{1}{2}\ln|2| = \infty$ Therefore $x = 1$ is not part of the interval of convergence. We conclude that the interval of converge is $-1 \le x < 1$, or in interval notation, $[-1, 1)$.

7. The following table is helpful:

day	amount right after 12:00 noon
Jan 1	2
Jan 2	$2 + (0.85)\ 2$
Jan 3	$2 + (0.85)\ 2 + (0.85)^2\ 2$
Jan 4	$2 + (0.85)\ 2 + (0.85)^2\ 2 + (0.85)^3\ 2$
$\vdots$	$\vdots$
Jan 20	$2 + (0.85)\ 2 + (0.85)^2\ 2 + \cdots + (0.85)^{19}\ 2$

Then, the amount right after noontime on January 20th is (note that there are 20 terms in the left-hand-side of equation):

$$2 + (0.85)\ 2 + (0.85)^2\ 2 + \cdots + (0.85)^{19}\ 2 = \frac{2(1-(0.85)^{20})}{1-0.85} \approx 12.8165$$

In the long run, the amount of food after noontime is $\frac{2}{1-0.85} \approx 13.3333$

8. a) Take $x = 0.5$, so the error is $E = f(0.5) - P_2(0.5) = 0.4055 - 0.3750 = 0.0305$

   b) The function $|f^{(3)}(t)| = |2/(1+t)^3|$ attains its maximum in the interval $0 \le t \le 0.5$ at $t = 0$. The maximum is $M = |f^{(3)}(0)| = 2$. The error bound when approximating $f(0.5)$ by $P_2(0.5)$ is $|f(0.5 - P_2(0.5)| \le M\frac{(0.5)^3}{3!} \approx 0.04166$

9. a) Begin from the binomial series with exponent $p = -1/2$:

$$(1+y)^{-1/2} = 1 + \frac{-1}{2}y + \frac{\frac{-1}{2}(\frac{-1}{2}-1)}{2!}y^2 + \frac{\frac{-1}{2}(\frac{-1}{2}-1)(\frac{-1}{2}-2)}{3!}y^3 + \cdots = 1 - \frac{y}{2} + \frac{3\,y^2}{8} - \frac{5\,y^3}{16} + \cdots, \quad -1 < y < 1.$$

   Substitute $y = t^2$ to obtain

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{t^2}{2} + \frac{3\,t^4}{8} - \frac{5\,t^6}{16} + \cdots$$

   The expansion given above is valid on $-1 < t < 1$.

   b) Since $\cos(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \cdots$ and $\frac{1}{\sqrt{1+t^2}} = 1 - \frac{t^2}{2} + \frac{3\,t^4}{8} - \frac{5\,t^6}{16} + \cdots$ we see that $\cos(t) < \frac{1}{\sqrt{1+t^2}}$. Reason: in the series expansions, the first terms that are different are the coefficients of t^2. Note that the coefficient of t^2 in the expansion of $\cos(t)$ is smaller than the coefficient of t^2 in the expansion of $\frac{1}{\sqrt{1+t^2}}$. (For small t, smaller powers dominate).

10. Short answer: $F_3(x) = \frac{1}{4} + \frac{3}{\pi}\sin(t) + \frac{1}{\pi}\sin(3t)$.

11. Short answer: $F_3(t) = 6\sin(t) - 3\sin(2t) + 2\sin(3t)$

12. (a) Comparison test: Note that $\frac{n^2}{1+n^3}$ behaves like $\frac{1}{n}$ when n is large, so we suspect that the series diverges. The following inequalities are clearly valid:

$$0 \leq \frac{n^2}{n^3+n^3} \leq \frac{n^2}{1+n^3}, \quad n = 2, 3, \ldots$$

The term in the center simplifies to $\frac{1}{2n}$. Since $\sum_{n=2}^{\infty} \frac{1}{2n}$ diverges, so does $\sum_{n=2}^{\infty} \frac{n^2}{1+n^3}$.

(b) Integral test: Set $f(x) = \frac{x^2}{1+x^3}$ for $x \geq 2$. A plot of $f(x)$ confirms that it is decreasing and positive for $2 \leq x \leq \infty$ as required by the test. Now $\int_2^{\infty} f(x)dx = \lim_{b\to\infty} \int_2^b \frac{x^2}{1+x^3}dx = \lim_{b\to\infty} \ln|1+x^3|_2^b = \lim_{b\to\infty} \frac{1}{3}\ln|1+b^3| - \frac{1}{3}\ln 9 = \infty$ Therefore, the series diverges too.

13. (a) Comparison test: Clearly,

$$0 \leq 1 \leq 1 + \frac{1}{n}, \quad n = 1, 2, \ldots$$

Since $\sum_{n=1}^{\infty} 1$ is divergent, so is our original series.

(b) Integral test: set $f(x) = 1+1/x$. Clearly $f(x)$ is decreasing and positive on $1 \leq x \leq \infty$ as required by the integral test. Now

$\int_1^{\infty}(1+1/x)dx = \lim_{b\to\infty} \int_1^b (1+1/x)dx = \lim_{b\to\infty} x + \ln(x)|_1^b = \lim_{b\to\infty}(b+\ln(b)) - (1+\ln(1)) = \infty.$

Since the integral diverges, the series diverges too.

14. (a) Geometric series:

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{\pi^n} = \sum_{n=0}^{\infty} -2\left(\frac{-2}{\pi}\right)^n = \text{a convergent geometric series,since } |x| = |-2/\pi| < 1$$

(b) Alternating series test: Since $\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{\pi^n} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(2)^{n+1}}{\pi^n}$ the series is of the alternating (sign) type. To apply the test we must first check two things. The first is,

$$\frac{2^1}{\pi^0} \geq \frac{2^2}{\pi^1} \geq \frac{2^3}{\pi^2} \geq \cdots$$

and this is clearly true. Also, we must check that

$$\lim_{n\to\infty} \frac{(2)^{n+1}}{\pi^n} = 0$$

This is clearly true once we rewrite the limit in this form:

$$\lim_{n\to\infty} 2\left(\frac{2}{\pi}\right)^n = 0$$

(reason: $2/\pi$ is less than 1, so raising this to a large power produces a small number). Because the two conditions to apply the test are satisfied, we have that, by the alternating series test, the series is convergent.

(c) Ratio test:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+2}}{\pi^{n+1}}}{\frac{(-2)^{n+1}}{\pi^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}2^{n+2}\pi^n}{(-1)^{n+1}2^{n+1}\pi^{n+1}} \right| = \lim_{n\to\infty} \frac{2}{\pi} = \frac{2}{\pi} \approx 0.6366$$

Since $L < 1$, the series converges.