9.4 Binomial Data: Testing $H_0: \; p_X = p_Y$

Consider experiments with two binomial sets of data corresponding to two treatments:

Treatment	# successes	# trials	P(success)
X	x	n	p_X
Y	y	m	p_Y

Problem:

at the α level of significance, consider

$$\boxed{\begin{array}{rc} H_0: & p_X = p_Y \\ & \text{vs.} \\ H_1: & p_X \neq p_Y \end{array}}$$

Observation:

Under H_0, we have $p_X = p_Y = p$ and

$$
\begin{aligned}
E\left(\frac{X}{n} - \frac{Y}{m}\right) &= \frac{E(X)}{n} - \frac{E(Y)}{m} \\
&= \frac{np}{n} - \frac{mp}{m} = 0 \\
Var\left(\frac{X}{n} - \frac{Y}{m}\right) &= \frac{Var(X)}{n^2} + \frac{Var(Y)}{m^2} \\
&= \frac{np(1-p)}{n^2} - \frac{mp(1-p)}{m^2} \\
&= p(1-p)\left(\frac{1}{n} + \frac{1}{m}\right)
\end{aligned}
$$

FACT: From the Central Limit Theorem, it follows that the following quantity has an approximate standard normal distribution:

$$\frac{\frac{X}{n} - \frac{Y}{m} - E\left(\frac{X}{n} - \frac{Y}{m}\right)}{\sqrt{\mathrm{Var}\left(\frac{X}{n} - \frac{Y}{m}\right)}} \approx Z$$

If, in addition, H_0 holds, then we may use the results in the previous slide to get

$$\frac{\frac{X}{n} - \frac{Y}{m}}{\sqrt{p(1-p)\left(\frac{1}{n} + \frac{1}{m}\right)}} \approx Z$$

Theorem 9.4.1 Let x and y denote the number of successes observed in two independent sets of n and m Bernoulli trials. Let

$$\hat{p} := \frac{x+y}{n+m} \quad \text{and} \quad z := \frac{\frac{x}{n} - \frac{y}{m}}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{n}+\frac{1}{m})}}$$

Test	Sig.	Action
$\begin{cases} H_0 : p_X = p_Y \\ H_1 : p_X > p_Y \end{cases}$	α	Reject H_0 if $z \geq z_\alpha$
$\begin{cases} H_0 : p_X = p_Y \\ H_1 : p_X < p_Y \end{cases}$	α	Reject H_0 if $z \leq -z_\alpha$
$\begin{cases} H_0 : p_X = p_Y \\ H_1 : p_X \neq p_Y \end{cases}$	α	Reject H_0 if $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$

Case Study 9.4.1

mitigation rate is defined as the proportion of cases where the defendant qualifies for prison time but gets a shortened prison term (due to plea bargains).

The mitigation rate in Escambia county, Fla. from January'94 to March'96 was 61.7% (1033 out of 1675 cases).

In April of 1996 the State Attorney established a new policy designed to limit plea bargains. From July'96 to June'97 mitigation decreased to 52.1% (344 out of 660 cases).

Is it fair to attribute the decrease to the new policy? Use $\alpha = 0.01$ to answer the question.

ANSWER: we wish to test

$$H_0 \;:\; p_X = p_Y = p \quad \text{vs.} \quad H_1 \;:\; p_X > p_Y$$

at the $\alpha = 0.01$ level. Note that

$$z_{0.01} = 2.33$$

We have,

$$\hat{p} = \frac{1033 + 344}{1675 + 660} = \frac{1377}{2335} = 0.590$$

and

$$x = \frac{1033}{1675} = 0.617, \quad y = \frac{344}{660} = 0.521,$$

so

$$z = \frac{0.617 - 5.21}{\sqrt{0.59(1 - 0.59)(\frac{1}{1675} + \frac{1}{660})}} = 4.25$$

Hence we reject the null hypothesis. The new policies seem to have an effect on lowering the mitigation rate.

Case Study 9.4.2 160 men and 192 women were asked whether they experienced nightmares often (at least once a month) or seldom (less than once a month). The results are given in the table:

	Men	Women	total
Nightmares often	55	60	115
Nightmares seldom	105	132	237
Totals	160	192	

Is the difference in the answers of men and women statistically significant at the 0.05 level?

ANSWER: Let p_M and p_W be the true proportions of men and women that have nightmares often. We wish to test

$$H_0 \; : \; p_M = p_W = p \quad \text{vs.} \quad H_1 \; : \; p_M \neq p_W$$

at the $\alpha = 0.05$ level. Note that

$$z_{0.025} = 1.96$$

We have,

$$\hat{p} = \frac{55+60}{160+192} = 0.327$$

and

$$x = \frac{55}{160} = 34.4, \quad y = \frac{60}{192} = 31.3,$$

so

$$z = \frac{0.344 - 0.313}{\sqrt{0.327(1-0.327)(\frac{1}{160} + \frac{1}{192})}} = 0.62$$

Then fail to reject the null hypothesis.

9.5 Confidence Intervals for the two-sample problem

Theorem 9.5.1 Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be independent random samples drawn from normal distributions with means μ_X and μ_Y respectively, and with the same standard deviation, σ. Let s_p denote the pooled standard deviation. A $100(1-\alpha)\%$ confidence interval for $\mu_X - \mu_Y$ is

$$\Big(\overline{x} - \overline{y} - t_{\alpha/2, n+m-2} \cdot s_p \cdot \sqrt{\tfrac{1}{n} + \tfrac{1}{m}},$$

$$\overline{x} - \overline{y} + t_{\alpha/2, n+m-2} \cdot s_p \cdot \sqrt{\tfrac{1}{n} + \tfrac{1}{m}}\Big)$$

The proof follows from Theorem 9.5.1.

Case Study 9.5.1 The numbers in the table show measures of rate of change of amount of X-ray penetration through a 500-micron section of tooth enamel at a wavelenght of 600 nm as opposed to 400 nm (8 males and 8 females).

Male, x_i	Female, y_i
4.9	4.8
5.4	5.3
5.0	3.7
5.5	4.1
5.4	5.6
6.6	4.0
6.3	3.6
4.3	5.0

Find a 95% condifence interval for $\mu_X - \mu_Y$.

Is 0 in the interval? What does this mean?

ANSWER: We have the following:

$$\sum x_\ell = 43.4, \quad \sum x_\ell^2 = 239.32, \quad \overline{x} = 5.2$$

$$s_X^2 = \frac{8(239.32) - (43.4)^2}{8 \cdot 7} = 0.55$$

$$\sum y_\ell = 36.1, \quad \sum y_\ell^2 = 166.95, \quad \overline{y} = 4.5$$

$$s_Y^2 = \frac{8(166.95) - (36.1)^2}{8 \cdot 7} = 0.58$$

$$s_p = \sqrt{\frac{7(0.55) + 7(0.58)}{8 + 8 - 2}} = 0.75$$

From the table we get $t_{0.025,14} = 2.1448$. Then, the 95% condifence interval is

$$\left(\overline{x} - \overline{y} - t_{\alpha/2,n+m-2} \cdot s_p \cdot \sqrt{\frac{1}{n} + \frac{1}{m}},\right.$$

$$\left.\overline{x} - \overline{y} + t_{\alpha/2,n+m-2} \cdot s_p \cdot \sqrt{\frac{1}{n} + \frac{1}{m}}\right)$$

$$= (0.1, 1.7)$$

Theorem 9.5.2 Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be independent random samples drawn from normal distributions with standard deviations σ_X and σ_Y.

A $100(1-\alpha)\%$ confidence interval for σ_X^2/σ_Y^2 is

$$\left(\frac{s_X^2}{s_Y^2} F_{\alpha/2,m-1,n-2}\ ,\ \frac{s_X^2}{s_Y^2} F_{1-\alpha/2,m-1,n-2}\right)$$

Case Study 9.5.2 Two sets of flow rates for Antartic's Hoseason Glacier have been calculated, one based on photos taken three years apart, the other, five years apart. On the basis of other considerations, it can be assumed that the true flow rate is constant.

3-year span	5 year span
0.73	0.72
0.76	0.74
0.75	0.74
0.77	0.72
0.73	0.72
0.75	
0.74	

Construct a 95% confidence interval for the variance ratio. Is "1" in the interval? What does this tell you?

ANSWER We have,

$$\sum x_\ell = 5.23, \quad \sum x_\ell^2 = 3.9089, \quad s_X^2 = 0.000224$$

$$\sum y_\ell = 3.64, \quad \sum y_\ell^2 = 2.6504, \quad s_Y^2 = 0.000120$$

From the table we get

$$F_{0.025,4,6} = 0.109, \quad F_{0.975,4,6} = 6.23$$

Substituting into the formula we have,

$$\left(\frac{s_X^2}{s_Y^2} F_{\alpha/2,m-1,n-2} \ , \ \frac{s_X^2}{s_Y^2} F_{1-\alpha/2,m-1,n-2} \right)$$

$$= (0.203, 11.629)$$

Theorem 9.5.3 Let x and y denote the numbers of successes observed in two independent sets of n and m Bernoulli trials respectively.

If p_X and p_Y denote the true success probabilities, an approximate $100(1-\alpha)\%$ confidence interval is given by

$$\left(\frac{x}{n} - \frac{y}{m} - z_{\alpha/2} \cdot \sqrt{\frac{1}{n}\frac{x}{n}(1-\frac{x}{n}) + \frac{1}{m}\frac{y}{m}(1-\frac{y}{m})}\ ,\right.$$

$$\left.\frac{x}{n} - \frac{y}{m} + z_{\alpha/2} \cdot \sqrt{\frac{1}{n}\frac{x}{n}(1-\frac{x}{n}) + \frac{1}{m}\frac{y}{m}(1-\frac{y}{m})}\right)$$

Case Study 9.5.3 Joseph Lister conjectured that human infections might have origin in yeast and bacterial infection. He began using carbolic acid as operating room disinfectant. The table show the mortality rates for 75 amputations performed by Lister, 35 without carbolic acid and 40 with it.

		C. Acid	Used ?	
		No	Yes	Total
Patient	Yes	19	34	53
lived?	No	16	6	22
	Total	35	40	

Let p_w and p_o resp. be the true survival probabilities for patients amputated "with" and "without" carbolic acid, respectively. Construct a 95% confidence interval for $p_w - p_o$. Is 0 in the interval? What does this tell you?

ANSWER We have,

$$x = 34, \quad n = 40, \quad y = 19, \quad m = 35$$

We also have $z_{0.025} = 1.96$

The 95% confidence interval is,

$$\left(\frac{x}{n} - \frac{y}{m} - z_{\alpha/2} \cdot \sqrt{\frac{1}{n}\frac{x}{n}(1 - \frac{x}{n}) + \frac{1}{m}\frac{y}{m}(1 - \frac{y}{m})}\ ,\right.$$

$$\left. \frac{x}{n} - \frac{y}{m} + z_{\alpha/2} \cdot \sqrt{\frac{1}{n}\frac{x}{n}(1 - \frac{x}{n}) + \frac{1}{m}\frac{y}{m}(1 - \frac{y}{m})} \right)$$

$$= (\ 0.31 - 1.96\sqrt{0.0103}\ ,\ 0.31 + 1.96\sqrt{0.0103}\ )$$

$$= (0.11, 0.51)$$