Chapter 9: Two-Sample Problems

It is quite common to have experiments where designed to compare responses to treatments with a control, or to compare several treatment levels. The two-sample problem falls into one of two formats: two treatments are applied to two independent but similar sets of subjects the same treatment is applied to two kinds of subject. Inference in two-sample problems is applied usually to comparison of means. The standard method is the Two-Sample test (section 9.2) But sometimes comparison of variabilities is sought (variance, standard deviation). A test based on the F distribution is used. (Section 9.3) A binomial version of the Two-sample problem will be discussed in Section 9.4. In section 9.5 we will discuss confidence intervals for the difference of means of two populations, and for the quotient of standard deviations.

A Preliminary Result

GIVEN:

$X_1, \dots, X_n$ = random sample from $\sim N(\mu_X, \sigma)$

$Y_1, \dots, Y_m$ = random sample from $\sim N(\mu_Y, \sigma)$

Let $\overline{X}$, $\overline{Y}$ be the corresponding sample means.

THEN

$$E(\overline{X}-\overline{Y}) = \mu_X - \mu_Y$$

$$Var(\overline{X}-\overline{Y}) = \frac{1}{n}\sigma^2 + \frac{1}{m}\sigma^2$$

$H_0: \ \mu_X = \mu_Y$ **the two-sample t-Test**

Problem 1: Test $H_0: \ \mu_X = \mu_Y$ when $\sigma_X = \sigma_Y$: EASY

Problem 2: Test $H_0: \ \mu_X \neq \mu_Y$ when $\sigma_X = \sigma_Y$: HARD

THEOREM 9.2.1 GIVEN:
$X_1, \dots, X_n$ = random sample from $\sim N(\mu_X, \sigma)$
$Y_1, \dots, Y_m$ = random sample from $\sim N(\mu_Y, \sigma)$
Let S_X^2, S_Y^2 = sample variances
Let S_p^2 = the pooled variance given by

$$S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2} = \frac{\sum_{\ell=1}^{n}(X_\ell - \overline{X}) + \sum_{\ell=1}^{m}(Y_\ell - \overline{Y})}{n+m-2}$$

$$\text{THEN} \quad t_{n+m-2} = \frac{\overline{X}-\overline{Y}-(\mu_X-\mu_Y)}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}$$

THEOREM 9.2.2 GIVEN:
$X_1, \dots, X_n$ = random sample from $\sim N(\mu_X, \sigma)$,
$Y_1, \dots, Y_m$ = random sample from $\sim N(\mu_Y, \sigma)$,
S_X^2, S_Y^2 = sample vars, S_p^2 = pooled var, $t = \frac{\bar{x}-\bar{y}}{s_p\sqrt{\frac{1}{n}+\frac{1}{m}}}$, THEN

Test	Sig.	Action
$H_0: \mu_X = \mu_Y$ $H_1: \mu_X > \mu_Y$	α	Reject H_0 if $t \geq t_{\alpha,n+m-2}$
$H_0: \mu_X = \mu_Y$ $H_1: \mu_X < \mu_Y$	α	Reject H_0 if $t \leq -t_{\alpha,n+m-2}$
$H_0: \mu_X = \mu_Y$ $H_1: \mu_X \neq \mu_Y$	α	Reject H_0 if $t \geq t_{\alpha/2,n+m-2}$ or $t \leq -t_{\alpha/2,n+m-2}$

Case Study 9.2.1

The followign table shows the proportion of 3-letter words in the works of two authors.

SEE TABLE 9.2.1

For the given data, we have

$$\overline{x} = 0.2319, \quad \overline{y} = 0.2097,$$

$$\sum_{\ell=1}^{8} x_\ell^2 = 0.4316, \quad \sum_{\ell=1}^{8} y_\ell^2 = 0.4406$$

Is the observed sample difference $\overline{x}-\overline{y}$ compatible with the statement that both authors are the same? —Use $\alpha = 0.01$.

ANSWER: We have,

$$s_X^2 = \frac{8 \cdot 0.4316 - (1.855)^2}{8 \cdot 7} = 0.0002103$$

$$s_Y^2 = \frac{10 \cdot 0.4406 - (2.097)^2}{10 \cdot 9} = 0.0000955$$

$$s_p = \sqrt{\frac{7 \cdot 0.0002103 + 9 \cdot 0.0000955}{8+10-2}} = \sqrt{0.0001457} = 0.0121$$

$$[\text{Theorem 9.2.1 \& } \mu_X = \mu_Y] \implies \frac{\overline{X}-\overline{Y}}{S_p\sqrt{\frac{1}{8}+\frac{1}{10}}} = t_{16}$$

With $\alpha = 0.01$ we get $t_{0.005,16} = 2.9208$,
so reject H_0 if $t \leq -2.9208$ or if $t \geq 2.9208$. But

$$t = \frac{0.2319 - 0.2097}{0.0121\sqrt{1/8+1/10}} = 3.88$$

Conclusion: Reject H_0.

Case Study 9.2.2

An instructor taught two sections of the same class using two different teaching strategies.
Students were asked to rate the "enthusiasm" of the instructor on a scale of 1 to 5.
The table shows the results from a student questionnaire given at the end of the semester.

Group I	Group II
n = 229	m = 243
$\bar{x} = 2.14$	$\bar{y} = 4.21$
$s_X = 0.94$	$s_Y = 0.83$

Set up a test to establish if the students perceived an increase in enthusiasm by the teacher. Use $\alpha = 0.05$.

ANSWER: Let μ_X and μ_Y be the means that correspond to the two teaching styles. Note a one sided-test is called for.

TEST: $H_0 : \mu_X = \mu_Y$ vs. $H_1 : \mu_X < \mu_Y$

# of degrees of freedom is $n + m - 2 = 229 + 243 - 2 = 470$

H_0 is rejected if $t < -t_{0.05,470} \approx -z_{0.05} = -1.64$

Finally, compute t by first finding s_p:

$$s_p = \sqrt{\frac{228(0.94)^2 + 242 * (0.83)^2}{229 + 243 - 2}} = 0.885$$

Then,

$$t = \frac{2.14 - 4.21}{0.885\sqrt{\frac{1}{229} + \frac{1}{243}}} = -25.42$$

so we REJECT H_0.

What if σ_X and σ_Y are known?

In this case, Thm 9.2.2 DOES NOT APPLY!!

Reason: if the X_ℓ's and the Y_ℓ's are normally distributed, then

$$Z = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

so the observed z ratio must be used.

Section 9.3:
The F-test for testing $H_0 : \sigma_X^2 = \sigma_Y^2$

We consider two sets of data, normally distributed, but with possibly different mean and different variance.

We wish to test whether the variances are equal or not.

A Preliminary Calculation

RECALL: if U, V are independent, χ^2 with m and n degrees of freedom, then $F_{m,n} = \frac{U/m}{V/n}$

RECALL: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

CONSIDER $X \sim N(\mu_X, \sigma_X)$, $Y \sim N(\mu_Y, \sigma_Y)$ THEN

$$\frac{S_X^2}{S_Y^2} = \frac{\frac{\sigma_Y^2}{m-1}\frac{(m-1)S_Y^2}{\sigma_Y^2}}{\frac{\sigma_X^2}{n-1}\frac{(n-1)S_X^2}{\sigma_X^2}}$$

So, if $\sigma_X^2 = \sigma_Y^2$ we have:

$$\frac{S_X^2}{S_Y^2} = \frac{U/(m-1)}{V/(n-1)} = F_{m-1,n-1}$$

THEOREM 9.3.1 GIVEN: (data with different variance)
$X_1, \dots, X_n$ = random sample from $\sim N(\mu_X, \sigma_X)$
$Y_1, \dots, Y_m$ = random sample from $\sim N(\mu_Y, \sigma_Y)$
THEN

Test	Sig.	Action
$H_0 : \sigma_X^2 = \sigma_Y^2$ $H_1 : \sigma_X^2 > \sigma_Y^2$	α	Reject H_0 if $s_Y^2/s_X^2 \le F_{\alpha,m-1,n-1}$
$H_0 : \sigma_X^2 = \sigma_Y^2$ $H_1 : \sigma_X^2 < \sigma_Y^2$	α	Reject H_0 if $s_Y^2/s_X^2 \ge F_{1-\alpha,m-1,n-1}$
$H_0 : \sigma_X^2 = \sigma_Y^2$ $H_1 : \sigma_X^2 \ne \sigma_Y^2$	α	Reject H_0 if $s_Y^2/s_X^2 \le F_{\alpha/2,m-1,n-1}$ or $s_Y^2/s_X^2 \ge F_{1-\alpha/2,m-1,n-1}$

Case Study 9.3.1 Measurements of alpha wave frequencies were taken from two sets of 10 inmates, one of which consisting of prisoners in solitary confinement.

Nonconfined, x_ℓ	Confined, y_ℓ
10.7	9.6
10.7	10.4
10.4	9.7
10.9	10.3
10.5	9.2
10.3	9.3
9.6	9.9
11.1	9.5
11.2	9.0
10.4	10.9

The data suggests a decrease in frequency for prisioners in solitary confinement, and also suggests an increase in variability for such people.
Use an F test to determine if the increase of variability is statistically significant at the $\alpha = 0.05$ level.

ANSWER: we wish to test

$$\sigma_X = \sigma_Y \quad vs. \quad \sigma_X \neq \sigma_Y$$

From the table, for $F_{9,9}$ with $\alpha = 0.05$:
$\Longrightarrow$ reject H_0 if $F < 0.248$ or if $F > 4.03$.
We have,

$$\sum x_\ell = 105.8, \quad \sum x_\ell^2 = 1121.26$$

$$\sum y_\ell = 97.8, \quad \sum y_\ell^2 = 959.7$$

$$s_X^2 = \frac{10(1121.26)-(105.8)^2}{10\cdot 9} = 0.21, \quad s_Y^2 = \frac{10(959.7)-(97.8)^2}{10\cdot 9} = 0.36$$

The observed F ratio is: $F = \frac{s_X^2}{s_Y^2} = \frac{0.36}{0.21} = 1.71$

CONCLUSION: Do not reject H_0.