7.2 Point Estimates for μ and σ^2.

The most important function in all of statistics:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(y-\mu)^2}$$

Recall the MLE estimates for a given random sample of size n:

$$\hat{\mu} = \overline{y} = \frac{1}{n} \sum_{\ell=1}^{n} y_\ell$$

and

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{\ell=1}^{n} (y_\ell - \overline{y})^2$$

Recall that $\hat{\sigma}^2$ is biased. In practice, the sample variance S^2 is used

$$S^2 = \frac{1}{n-1} \sum_{\ell=1}^{n} (Y_\ell - \overline{Y})^2$$

Theorem 7.2.1 Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then $\overline{Y}$, the MLE for μ, is unbiased, efficient, and consistent. If in addition σ^2 is known, then $\overline{Y}$ is sufficient.

Here we prove consistency, that is, $\lim_{n\to\infty} P\left(\left|\overline{Y}_n - \mu\right|\right) = 1$. Chebyshev's inequality gives

$$P\left(\left|\overline{Y}_n - \mu\right| < \epsilon\right) \geq 1 - \frac{Var(\overline{Y})}{\epsilon^2}$$

Since $Var(\overline{Y}) = \sigma^2/n$, we have

$$1 - \frac{\sigma^2}{n\epsilon^2} \leq P\left(\left|\overline{Y}_n - \mu\right| < \epsilon\right) \leq 1$$

Now take limit on all three terms. The inequalities are preserved, and the outside limits give 1. By the "squeeze theorem" the middle limit is also 1.

Fitting a Normal Distribution to Data

GIVEN: n data points $y_1, \ldots, y_n$

GROUP data points into k equal-width classes

SET:

m_i = midpoint of i-th class

f_i = number of observations in i-th class.

COMPUTE the density for each class

$$\text{class density} = \frac{\text{class frequency}}{\text{class width} \times n}$$

or

$$d_i = \frac{f_i}{\Delta \times n}$$

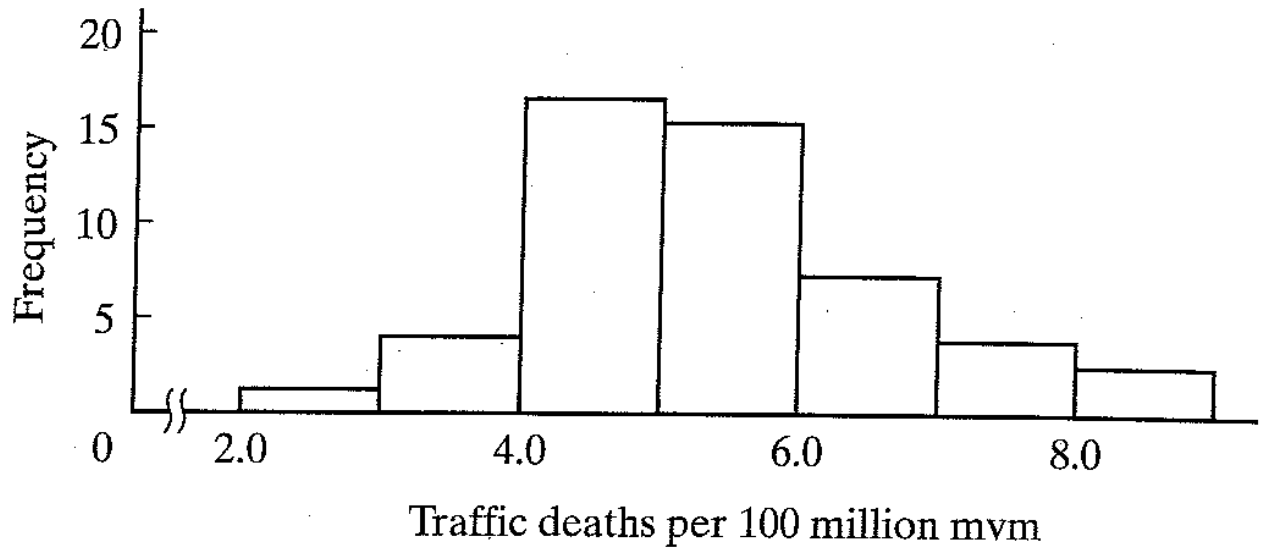

FIGURE 2.5.8

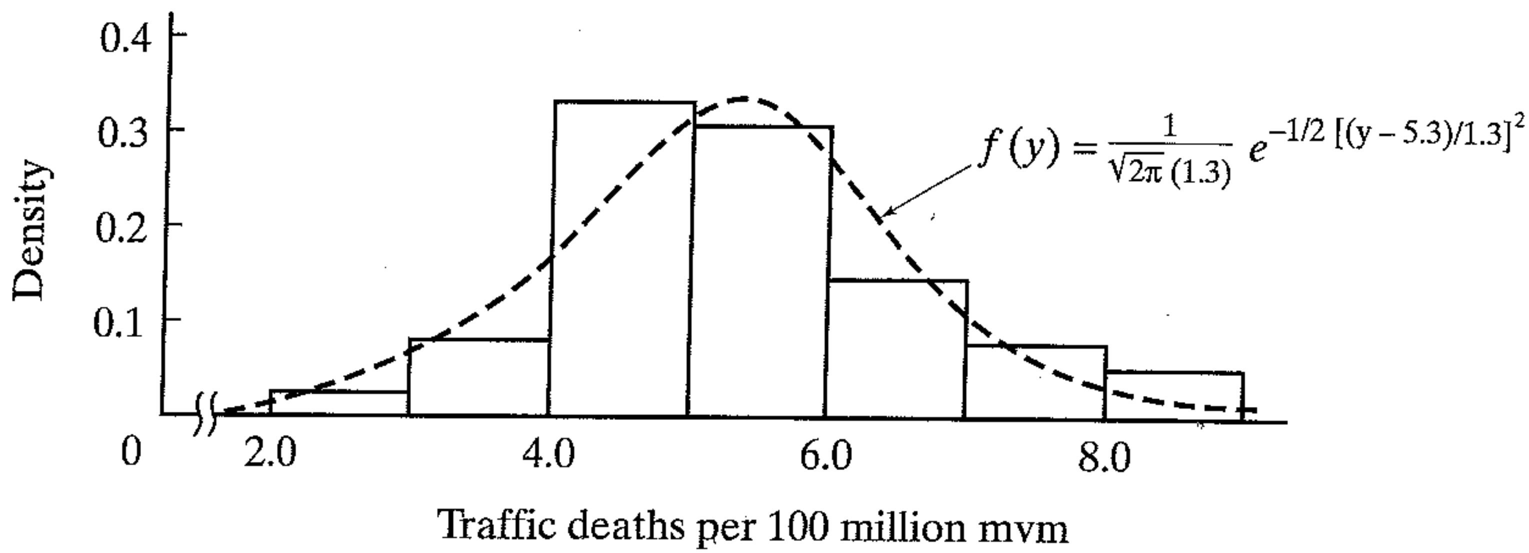

FIGURE 2.5.9

To estimate μ use the grouped sample mean

$$\overline{y}_g = \frac{1}{n}\sum_{i=1}^{k} f_i m_i$$

To estimate σ^2 use the grouped sample variance

$$\begin{aligned} s_g^2 &= \frac{1}{n-1}\sum_{i=1}^{k} f_i (m_i - \overline{y}_g)^2 \\ &= \frac{1}{n(n-1)}\left(n\sum_{i=1}^{k} f_i m_i^2 - \left(\sum_{i=1}^{k} f_i m_i\right)^2\right) \end{aligned}$$

In the previous example,

$$\overline{y}_g = 67.60 \quad \text{and} \quad s_g^2 = 6.656$$

Height of US Army recruits. Is it normally distributed?

TABLE 7.2.1

Height (in.)	Midpoint, m_i	Frequency, f_i	Density
$54.0 \le y < 55.0$	54.5	5	0.00019
$55.0 \le y < 56.0$	55.5	3	0.00012
$56.0 \le y < 57.0$	56.5	7	0.00027
$57.0 \le y < 58.0$	57.5	6	0.00023
$58.0 \le y < 59.0$	58.5	10	0.00039
$59.0 \le y < 60.0$	59.5	15	0.00058
$60.0 \le y < 61.0$	60.5	50	0.00193
$61.0 \le y < 62.0$	61.5	526	0.02033
$62.0 \le y < 63.0$	62.5	1237	0.04780
$63.0 \le y < 64.0$	63.5	1947	0.07524
$64.0 \le y < 65.0$	64.5	3019	0.11666
$65.0 \le y < 66.0$	65.5	3475	0.13428
$66.0 \le y < 67.0$	66.5	4054	0.15666
$67.0 \le y < 68.0$	67.5	3631	0.14031
$68.0 \le y < 69.0$	68.5	3133	0.12107
$69.0 \le y < 70.0$	69.5	2075	0.08018
$70.0 \le y < 71.0$	70.5	1485	0.05738
$71.0 \le y < 72.0$	71.5	680	0.02628
$72.0 \le y < 73.0$	72.5	343	0.01325
$73.0 \le y < 74.0$	73.5	118	0.00456
$74.0 \le y < 75.0$	74.5	42	0.00162
$75.0 \le y < 76.0$	75.5	9	0.00035
$76.0 \le y < 77.0$	76.5	6	0.00023
$77.0 \le y < 78.0$	77.5	2	0.00008
		25878	

Height of Army recruits. Superimposed normal curve.

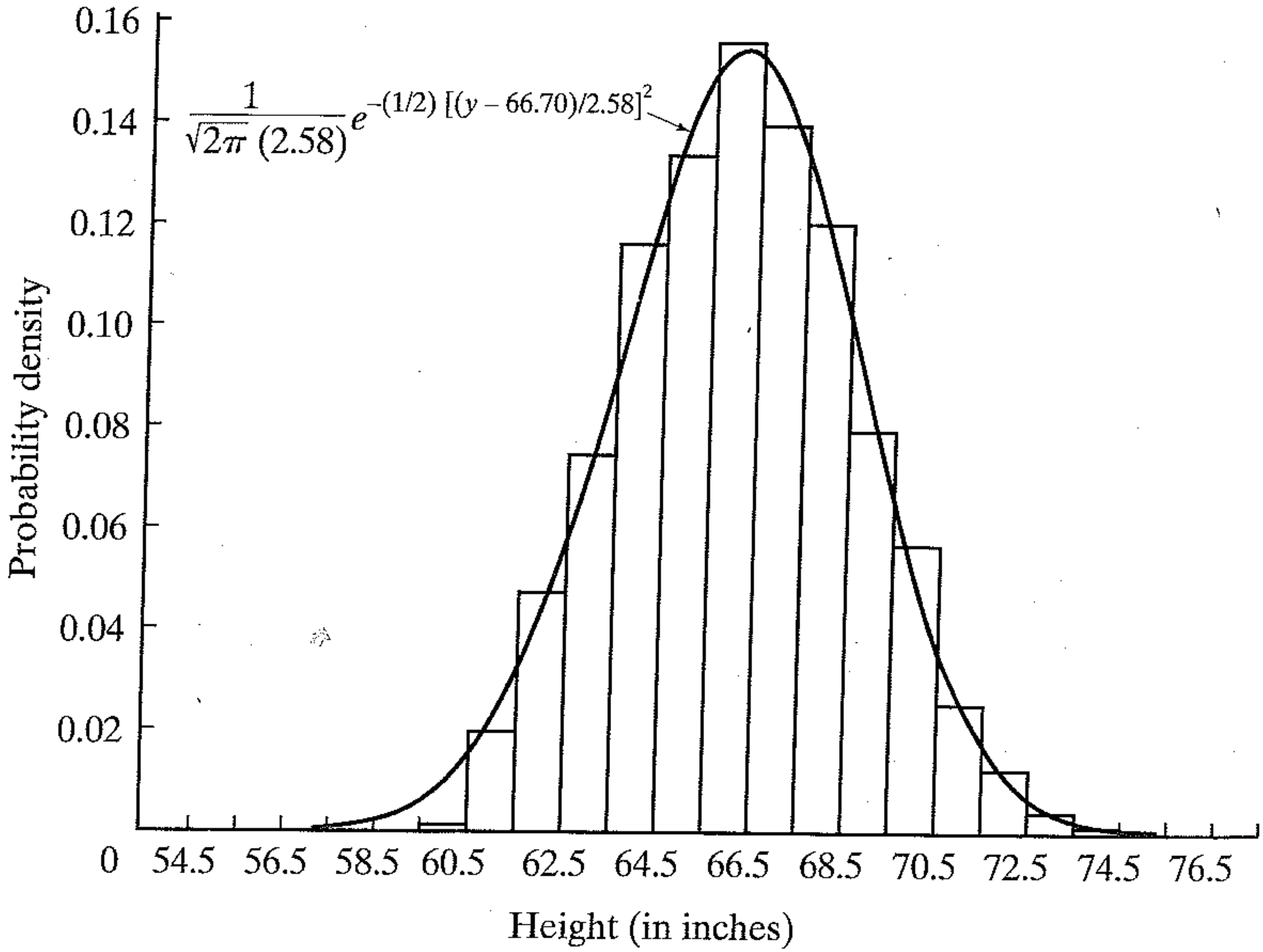

FIGURE 7.2.1

Drawing inferences about μ

Case Study 7.2.2: $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ vs. $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

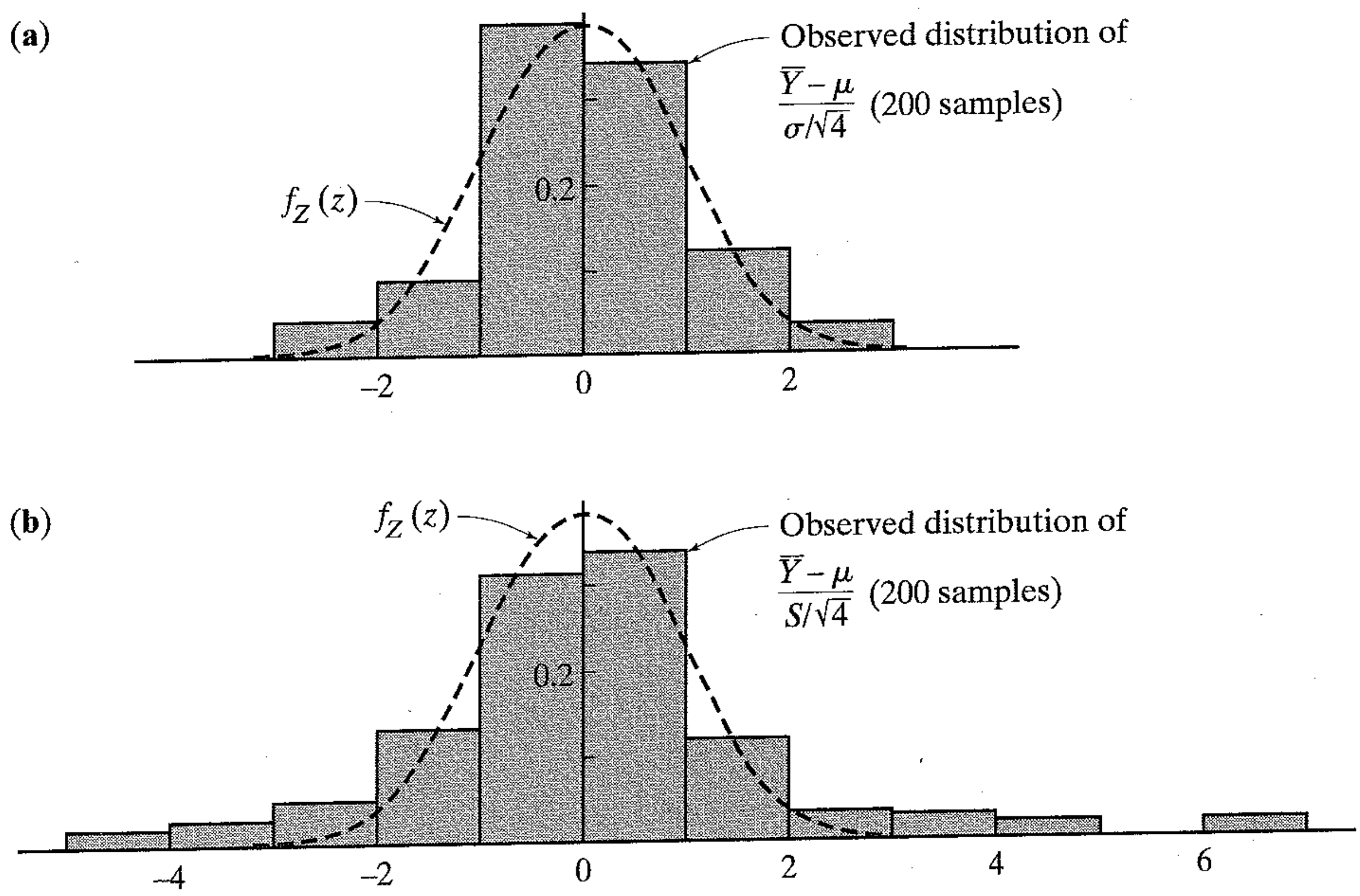

FIGURE 7.2.2