MTH 244 - Additional Problems for §1.5

Section 1 (Merino) and section 3 (Dobrushkin) - January 2003

Problems.

1. Given $\psi(x,y)$, find the exact differential equation $d\psi(x,y) = 0$.

a) $\psi(x,y) = x^2 + y^2$	f) $\psi(x,y) = \exp(xy^2)$
b) $\psi(x,y) = \ln(x^2y^2)$	g) $\psi(x,y) = (x+y-2)^2$
c) $\psi(x,y) = \tan(9x^2 + y^2)$	h) $\psi(x,y) = \sinh(x^3 - y^3)$
d) $\psi(x,y) = x/y^2$	i) $\psi(x,y) = \sin(x^2y)$
e) $\psi(x,y) = x^2y + 2y^2$	j) $\psi(x,y) = x + 3xy^2 + x^2y$.

2. Show that the following differential equations are exact and solve them

   (a) $x^2\, y' + 2yx = 0$
   (b) $y^3\, dx + 3xy^2\, dy = 0$
   (c) $(2x + e^y)dx + xe^y\, dy = 0$
   (d) $((2xy - 3x^2)\, dx + (x^2 + y)\, dy = 0$
   (e) $(\cos\theta - 2r\cos^2\theta)\, dr + r\sin\theta(2r\cos\theta - 1)\, d\theta = 0$
   (f) $(2r + \sin\theta - \cos\theta)\, dr + r(\sin\theta + \cos\theta)\, d\theta = 0$
   (g) $2x(1 + \sqrt{x^2 - y})\, dx = \sqrt{x^2 - y}\, dy$
   (h) $(1 - xy)^{-2}\, dx + [y^2 + x^2(1 - xy)^{-2}]\, dy = 0$
   (i) $2xy\, dx + (x^2 + 4y)\, dy = 0$
   (j) $(\cos xy - \sin xy)\,(y\, dx + x\, dy) = 0$
   (k) $y\ (e^{xy} + y)\ dx + x\ (e^{xy} + 2y)\ dy = 0$
   (l) $(y/x + x)\, dx + (y\ \ln x - 1)\, dy = 0$
   (m) $e^{-\theta}\, dr - r\, e^{-\theta}\, d\theta = 0$
   (n) $(\cot y + x^3)\, dx = x\ \csc^2 y\, dy$
   (o) $(6xy + 2y^2 - 4)\, dx + (3x^2 + 4xy - 1)\, dy = 0.$

3. Solve the following exact equations:

a) $2x\left[3x^2 + y - ye^{-x^2}\right]\, dx + \left[x^3 + 3y^2 + e^{-x^2}\right]\, dy = 0$	f) $\frac{y}{x}\, dx + (y^2 + \ln x)\, dy = 0$
b) $\left(\frac{1}{y} + \frac{y}{x^2}\right)\, dx - \left(\frac{1}{x} + \frac{x}{y^2}\right)\, dy = 0$	g) $\frac{3x^2+y^2}{y^2}\, dx - \frac{2x^3+4y}{y^3}\, dy = 0$
c) $\left(\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right)\, dx - \frac{x}{(y-x)^2}\, dy = 0$	h) $\ln y\ \sinh x\, dx + \frac{\cosh x}{y}\, dy = 0$
d) $\left(\frac{y}{x} + y + \frac{1}{y}\right)\, dx + \left(x + y - \frac{1}{x} - \frac{x}{y^2}\right)\, dy = 0$	i) $\frac{y\, dx + x\, dy}{1+x^2y^2} = 0$
e) $\left[y^2 - \frac{1}{x^2y}\right]\, dx + \left[3xy^2 - \frac{1}{xy^2} - 2\right]\, dy = 0.$	j) $\left(y - \frac{3}{x}\right)\, dx + \left(x - \frac{3}{y} + 1\right)\, dy = 0.$

4. Are the following equations exact? Solve the initial value problems

   (a) $\cos\pi x\ \cos 2\pi y\, dx = 2\sin\pi x\ \sin 2\pi y\, dy,\ \ y(3/2) = 1/2$
   (b) $2xy\, dy = (x^2 + y^2)\, dx,\ \ y(2) = 4$
   (c) $(2xy - 4)\, dx + (x^2 + 4y - 1)\, dy = 0,\ \ y(1) = 2,$
   (d) $\sin\omega y\, dx + \omega x\cos\omega y\, dy = 0,\ \ y(1) = \pi/2\omega$
   (e) $(y - 1)\, dx + (x - 2)\, dy = 0,\ \ y(1) = 3$
   (f) $4x^{-1}\, dx + dy = 0,\ \ y(1) = 4$
   (g) $x^{-1}e^{-y/x}\, dy - x^{-2}ye^{-y/x}\, dx = 0,\ \ y(-2) = -2$

(h) $(2x - 3y)\,dx + (2y - 3x)\,dy = 0,\ y(0) = 1$

(i) $3y(x^2 - 1)\,dx + (x^3 + 8y - 3x)\,dy = 0,\ y(0) = 1$

(j) $(xy^2 + x - 2y + 5)\,dx + (x^2y + y^2 - 2x)\,dy = 0,\ y(1) = 2$

(k) $2y^2 \sin^2 x\,dx - y \sin 2x\,dy = 0,\ y(\pi/2) = 1$

(l) $2x^2\,dx - 3y^3\,dy = 0, \quad y(1) = 0$

(m) $e^y\,dx - (2x + y)\,dy = 0, \quad y(1) = 2$

(n) $(\sin y + y \cos x)\,dx + (\sin x + x \cos y)\,dy = 0, \quad y(\pi) = 2$

(o) $(\sin xy + xy \cos xy)\,dx + x^2 \cos xy\,dy = 0, \quad y(0) = 1$

(p) $(3x^2 \sin 2y - 2xy)\,dx + (2x^3 \cos 2y - x^2)\,dy = 0, \quad y(1) = 0$

(q) $(xy^2 + y)\,dx + (x^2y + x - 3y^2)\,dy = 0, \quad y(0) = -1$

(r) $(y^4 - 3y + 4x)\,dx + (4xy^3 + 3y^2 - 3x)\,dy = 0, \quad y(1) = 1$

5. If the differential equation $M\,dx + N\,dy = 0$ is exact then prove that

$$[M(x, y) + f(x)]\,dx + [N(x, y) + g(y)]\,dy = 0$$

is also exact for any differentiable functions $f(x)$ and $g(y)$.

6. Show that the linear fractional equation

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

is exact if and only if $A + b = 0$.

7. In each of the following equations determine the constant λ such that the equation is exact, and solve the resulting exact equation:

a) $(x^2 + 3xy)\,dx + (\lambda x^2 + 4y - 1)\,dy = 0,$
b) $\left(\frac{1}{x^2} + \frac{1}{y^2}\right)\,dx + \frac{\lambda x + 1}{y^3}\,dy = 0,$
c) $(\lambda x^2 y + 2y^2 - 1)\,dx + (x^3 + 4xy + 4)\,dy = 0,$
d) $(e^x \sin y + 3y)\,dx + (\lambda e^x \sin y - 3x)\,dy = 0,$
e) $(2x + \lambda y^2)\,dx - 2xy\,dy = 0,$
f) $\left(\frac{\lambda y}{x^3} + \frac{y}{x^2}\right)\,dx + \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dy = 0,$
g) $\lambda xy + y^2)\,dx + (x^2 + y^2)\,dy = 0,$
h) $\left(\frac{y}{x} + 6x\right)\,dx + (\ln x + \lambda)\,dy = 0.$

8. In each of the following equations determine the most general function $M(x, y)$ such that the equation is exact:

a) $M(x, y)\,dx + (2xy - 4)\,dy = 0,$
b) $M(x, y)\,dx + (2x + y)\,dy = 0,$
c) $M(x, y)\,dx + (3x^2 + 4xy - 6)\,dy = 0,$
d) $M(x, y)\,dx + \sqrt{3x^2 - y}\,dy = 0,$
e) $M(x, y)\,dx + (2x^2 - y^2)\,dy = 0,$
f) $M(x, y)\,dx + 4y^3 - 2x^3y)\,dy = 0,$
g) $M(x, y)\,dx + (y^3 + \ln x)\,dy = 0,$
h) $M(x, y)\,dx + 2y\,\sin x\,dy = 0.$

9. In each of the following equations determine the most general function $N(x, y)$ such that the equation is exact:

$(2xy + 1)\,dx + N(x, y)\,dy = 0,$
$(e^y + x)\,dx + N(x, y)\,dy = 0,$
$(2 + y^2 \cos 2x)\,dx + N(x, y)\,dy = 0,$
$x\,e^{-y}\,dx + N(x, y)\,dy = 0,$
$(3x + 2y)\,dx + N(x, y)\,dy = 0,$
$(2y^{3/2} + 1)\,x^{-1/2}\,dx + N(x, y)\,dy = 0,$
$(3x^2 + 2y)\,dx + N(x, y)\,dy = 0,$
$2xy^{-3} + N(x, y)\,dy = 0,$