Solve the following initial value problems.

28. $y' = -\dfrac{x + 2y}{y}$
$y(1) = 1$

29. $(x^2 + y^2)dx - 2xy\,dy = 0$
$y(1) = 1$

30. $y' = \sqrt{\dfrac{x + y}{2x}}$
$y(1) = 2$

31. $y' = \dfrac{2x + y - 4}{x - y + 1}$
$y(2) = 2$

1.7 EQUATIONS REDUCIBLE TO FIRST ORDER

In this section we study two types of higher-order ordinary differential equations that can be reduced to first-order equations by means of simple transformations.

A. Equations of the form

$$y^{(n)} = F(x, y^{(n-1)}), \tag{1}$$

containing only two consecutive derivatives $y^{(n)}$ and $y^{(n-1)}$ can be reduced to first order by means of the transformation

$$w = y^{(n-1)}. \tag{2}$$

In fact, differentiating both sides of Eq. (2) with respect to x, we find $w' = y^{(n)}$, and using (1), we obtain

$$w' = F(x, w).$$

EXAMPLE 1 Compute the general solution of the differential equation

$$y''' - \frac{1}{x}y'' = 0. \tag{3}$$

Solution Setting $w = y''$, Eq. (3) becomes

$$w' - \frac{1}{x}w = 0. \tag{4}$$

Equation (4) is separable (and linear) with general solution

$$w(x) = c_1 x.$$

Thus, $y'' = c_1 x$. Integrating with respect to x, we obtain $y' = \frac{1}{2}c_1x^2 + c_2$. Another integration yields $y = \frac{1}{6}c_1x^3 + c_2x + c_3$. As c_1 is an arbitrary constant, the general solution of Eq. (3) is

$$y(x) = c_1x^3 + c_2x + c_3. \tag{5}$$

REMARK 1 The general solution of an ordinary differential equation of order n contains n arbitrary constants. For example, Eq. (3) is of order 3, and as we have seen, its general solution (5) contains the three arbitrary constants c_1, c_2, and c_3.

B. Second-order differential equations of the form

$$y'' = F(y, y') \tag{6}$$

(they should not contain x) can be reduced to first order by means of the transformation

$$w = y'. \tag{7}$$

In fact, from (7) we obtain (using the chain rule)

$$y'' = \frac{dw}{dx} = \frac{dw}{dy}\cdot\frac{dy}{dx} = \frac{dw}{dy}w.$$

Thus, Eq. (6) becomes

$$w\frac{dw}{dy} = F(y, w),$$

which is first order with independent variable y and unknown function w. Sometimes this latter equation can be solved by one of our earlier methods.

EXAMPLE 2 Solve the IVP

$$y'' = y'(y' + y) \tag{8}$$
$$y(0) = 0$$
$$y'(0) = -1.$$

Solution Here the differential equation (8) does not contain x and therefore can be reduced to a first-order differential equation by means of the transformation $w = y'$. In fact, Eq. (8) becomes

$$\frac{dw}{dy} - w = y, \tag{9}$$

which is linear. Multiplying both sides of (9) by e^{-y} we obtain $(d/dy)(we^{-y}) = ye^{-y}$. Integrating with respect to y we find $we^{-y} = -ye^{-y} - e^{-y} + c_1$, and so

$$y'(x) = w = -y - 1 + c_1e^y.$$

Using the initial conditions, we find that $c_1 = 0$. Thus,

$$y' = -y - 1$$

and $y(x) = -1 + ce^{-x}$. Using the initial condition $y(0) = 0$, we find that $c =$., and the solution of Eq. (8) is

$$y(x) = -1 + e^{-x}.$$

APPLICATION 1.7.1

■ Assume that two compartments A_1 and A_2, of volumes V_1 and V_2, respecti
are separated by a barrier. Through the barrier a solute can diffuse from
compartment to the other at a rate proportional to the difference c_1 –
concentration of the two compartments, from the higher concentration to
lower. Find the concentration in each compartment at any time t.

Solution Let $y_1(t)$ and $y_2(t)$ be the amount of the solute in the compartm
A_1 and A_2, respectively, at time t. Then

$$c_1(t) = \frac{y_1(t)}{V_1} \quad \text{and} \quad c_2(t) = \frac{y_2(t)}{V_2}$$

are the concentrations in compartments A_1 and A_2, respectively. The differe
equations describing the diffusion are

$$\dot{y}_1(t) = k(c_2 - c_1) \quad \text{and} \quad \dot{y}_2 = k(c_1 - c_2),$$

where k is a constant of proportionality ($k > 0$). Dividing by the volumes o
compartments, we get

$$\dot{c}_1 = \frac{k}{V_1}(c_2 - c_1) \quad \text{and} \quad \dot{c}_2 = \frac{k}{V_2}(c_1 - c_2) = -\frac{\dot{y}_1}{V_2} = -\frac{V_1}{V_2}\dot{c}_1.$$

Differentiating both sides of the first equation and using the second equat
we find that

$$\ddot{c}_1 = \frac{k}{V_1}(\dot{c}_2 - \dot{c}_1) = -k\left(\frac{1}{V_1} + \frac{1}{V_2}\right)\dot{c}_1$$

or

$$\ddot{c}_1 + k\left(\frac{1}{V_1} + \frac{1}{V_2}\right)\dot{c}_1 = 0. \quad ($$

Equation (11) is a differential equation of form A and can be reduced to a fi
order differential equation. We leave the computational details for Exerc
5, 6, and 7.

EXERCISES

Compute the general solution of the following differential equations.

1. $y'' + y' = 3$

2. $y^{(5)} - y^{(4)} = 0$

Solve the following initial value problems.

3. $y'' = \dfrac{1 + y'^2}{2y}$, $y(0) = 1$, $y'(0) = -1$

4. $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$

In Exercises 5 through 7, let $c_1(0)$ and $c_2(0)$ be the initial concentrations in the two compartments in the chemistry application. Show that:

5. $c_1(t) = c_1(0) - \dfrac{V_2}{V_1 + V_2}[c_2(0) - c_1(0)]\left\{\exp\left[-k\left(\dfrac{1}{V_1} + \dfrac{1}{V_2}\right)t\right] - 1\right\}$

6. $c_2(t) = c_2(0) - \dfrac{V_1}{V_1 + V_2}[c_1(0) - c_2(0)]\left\{\exp\left[-k\left(\dfrac{1}{V_1} + \dfrac{1}{V_2}\right)t\right] - 1\right\}$

7. $c_1(\infty) = c_2(\infty) = \dfrac{V_1c_1(0) + V_2c_2(0)}{V_1 + V_2}$. That is, after a long time, the concentrations in the two compartments are equal, and so equilibrium would be reached.

Find the general solution of the following differential equations.

8. $xy'' + y' = 3$

9. $y^{(5)} - \dfrac{1}{x}y^{(4)} = 0$

10. $y''' + y'' = 1$

11. $y'' - y = 0$

12. $y'' = 2y' + 2y'y$

13. $y'' - 2y^{-3}y' = 0$

Astrophysics Differential equations of the form

$$y'' = (1 + y'^2)f(x, y, y')$$

have been obtained in connection with the study of orbits of satellites.[24] Solve this differential equation in each of the following cases.

14. $f(x, y, y') = \dfrac{1}{y}$

15. $f(x, y, y') = y'$

16. $f(x, y, y') = 1$

17. Solve the essentially "circular"[25] differential equation

$$y''' = y'(3y'' - y'''y')$$

(*Hint:* Rewrite the differential equation in the form

$$\frac{y'''(1 + y'^2) - 3y'y''}{(1 + y'^2)^{5/2}} = 0$$

and observe that the left-hand side is the derivative of the curvature

$$K = \frac{y''}{(1 + y'^2)^{3/2}}$$

of the solution curves.)

[24] See *Notices AMS* (Jan. 1975): A142.
[25] *Amer. Math. Monthly* 92 (1985): 511