INTEGRATING FACTORS
FOR SOLVING DEs THAT CAN
BE REDUCED TO "EXACT"

SOURCE: ODEs with modern applications
By Finizio-Ladas, 3rd ed.
Pages 52-53.

25. Integrating Factors If the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{25}$$

is not exact, that is, $M_y \neq N_x$, we can sometimes find a (nonzero) function μ that depends on x or y or both x and y such that the differential equation

$$\mu M\,dx + \mu N\,dy = 0 \tag{26}$$

is exact, that is, $(\mu M)_y = (\mu N)_x$. The function μ is then called an *integrating factor* of the differential equation (25). Since (26) is exact, we can solve it, and its solutions will also satisfy the differential equation (25).

Show that a function $\mu = \mu(x, y)$ is an integrating factor of the differential equation (25) if and only if it satisfies the partial differential equation

$$N\mu_x - M\mu_y = (M_y - N_x)\mu. \tag{27}$$

26. In general, it is very difficult to solve the partial differential equation (27) without some restrictions on the functions M and N of Eq. (25). In this and the following exercise the restrictions imposed on M and N reduce Eq. (27)

into a first-order linear differential equation whose solutions can be found explicitly. If it happens that the expression

$$\frac{1}{N}(M_y - N_x)$$

is a function of x alone, it is always possible to choose μ as a function of x only. Show that with these assumptions the function

$$\mu(x) = e^{\int (1/N)(M_y - N_x)dx}$$

is an integrating factor of the differential equation $M\,dx + N\,dy = 0$.

27. If it happens that the expression

$$\frac{1}{M}(M_y - N_x)$$

is a function of y alone, it is always possible to choose μ as a function of y only. Show that with these assumptions the function

$$\mu(y) = e^{-\int (1/M)(M_y - N_x)dy}$$

is an integrating factor of the differential equation $M\,dx + N\,dy = 0$.

For each of the following differential equations, find an integrating factor and then use it to solve the differential equation. (*Hint:* Use Exercise 26 or 27.)

28. $y\,dx - x\,dy = 0$

29. $(x^2 - 2y)dx + x\,dy = 0$

30. $y\,dx + (2x - y^2)dy = 0$

31. $(y - 2x)dx - x\,dy = 0$

32. $y\,dx - (x - 2y)dy = 0$

33. $(x^4 + y^4)dx - xy^3dy = 0$

34. $(x^2 - y^2 + x)dx + 2xy\,dy = 0$

35. Verify that $\mu(x) = e^{\int a(x)dx}$ is an integrating factor of the first-order linear differential equation

$$y' + a(x)y = b(x),$$

and then use it to find its solution. (*Hint:* Write the differential equation in the equivalent form $[a(x)y - b(x)]dx + dy = 0$.)

Verify that each of the following functions is an integrating factor of the differential equation $y\,dx - x\,dy = 0$ and then use the function to solve the equation.

36. $\mu(y) = \dfrac{1}{y^2}$ for $y \neq 0$

37. $\mu(x, y) = \dfrac{1}{xy}$ for $x \neq 0$ and $y \neq 0$

38. $\mu(x, y) = \dfrac{1}{x^2 + y^2}$ for $x \neq 0$ or $y \neq 0$