Elementary Methods—First-Order Differential Equations

1.1 INTRODUCTION AND DEFINITIONS

Differential equations are equations that involve derivatives of some unknown function(s). Although such equations should probably be called "derivative equations," the term "differential equations" (*aequatio differentialis*) initiated by Leibniz in 1676 is universally used. For example,

$$y' + xy = 3 \tag{1}$$

$$y'' + 5y' + 6y = \cos x \tag{2}$$

$$y'' = (1 + y'^2)(x^2 + y^2) \tag{3}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{4}$$

are differential equations. In Eqs. (1)–(3) the unknown function is represented by y and is assumed to be a function of the single independent variable x, that is, $y = y(x)$. The argument x in $y(x)$ (and its derivatives) is usually suppressed for notational simplicity. The terms y' and y'' in Eqs. (1)–(3) are the first and second derivatives, respectively, of the function $y(x)$ with respect to x. In Eq. (4) the unknown function u is assumed to be a function of the two independent variables t and x, that is, $u = u(t, x)$, $\partial^2 u/\partial t^2$ and $\partial^2 u/\partial x^2$ are the second partial derivatives of the function $u(t, x)$ with respect to t and x, respectively. Equation (4) involves partial derivatives and is a *partial differential equation*. Equations (1)–(3) involve ordinary derivatives and are *ordinary differential equations*.

In this book we are primarily interested in studying ordinary differential equations.

DEFINITION 1

An ordinary differential equation of order n is an equation that is, or can be put, in the form

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where $y, y', \ldots, y^{(n)}$ *are all evaluated at x.*

The independent variable x belongs to some interval I (I may be finite or infinite), the function F is given, and the function $y = y(x)$ is unknown. For the most part the functions F and y will be real valued. Thus, Eq. (1) is an ordinary differential equation of order 1 and Eqs. (2) and (3) are ordinary differential equations of order 2.

DEFINITION 2

A solution of the ordinary differential equation (5) *is a function* $y(x)$ *defined over a subinterval* $J \subset I$ *which satisfies Eq.* (5) *identically over the interval* J.

Clearly, any solution $y(x)$ of Eq. (5) should have the following properties:

1. y should have derivatives at least up to order n in the interval J.
2. For every x in J the point $(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$ should lie in the domain of definition of the function F, that is, F should be defined at this point.
3. $y^{(n)}(x) = F(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$ for every x in J.

As an illustration we note that the function $y(x) = e^x$ is a solution of the second-order ordinary differential equation $y'' - y = 0$. In fact.

$$y''(x) - y(x) = (e^x)'' - e^x = e^x - e^x = 0.$$

Clearly, e^x is a solution of $y'' - y = 0$ valid for all x in the interval $(-\infty, +\infty)$. As another example, the function $y(x) = \cos x$ is a solution of $y'' + y = 0$ over the interval $(-\infty, +\infty)$. Indeed,

$$y''(x) + y(x) = (\cos x)'' + \cos x = -\cos x + \cos x = 0.$$

In each of the illustrations the solution is valid on the whole real line $(-\infty, +\infty)$. On the other hand, $y = \sqrt{x}$ is a solution of the first-order ordinary differential equation $y' = 1/2y$ valid only in the interval $(0, +\infty)$ and $y = \sqrt{x(1-x)}$ is a solution of the ordinary differential equation $y' = (1-2x)/2y$ valid only in the interval $(0, 1)$.

As we have seen, $y = e^x$ is a solution of the ordinary differential equation $y'' - y = 0$. We further observe that $y = e^{-x}$ is also a solution and moreover $y = c_1 e^x + c_2 e^{-x}$ is a solution of this equation for arbitrary values of the constants c_1 and c_2. It will be shown in Chapter 2 that $y = c_1 e^x + c_2 e^{-x}$ is the "general solution" of the ordinary differential equation $y'' - y = 0$. By the general solution we mean a solution with the property that any solution of $y'' - y = 0$ can be obtained from the function $c_1 e^x + c_2 e^{-x}$ for some special values of the constants c_1 and c_2. Also, in Chapter 2 we will show that the general solution of the ordinary differential equation $y'' + y = 0$ is given by $y(x) = c_1 \cos x + c_2 \sin x$ for arbitrary values of the constants c_1 and c_2.

In this chapter we present elementary methods for finding the solutions of some first-order ordinary differential equations, that is, equations of the form

$$y' = F(x, y), \tag{6}$$

together with some interesting applications.

The differential of a function $y = y(x)$ is by definition given by $dy = y'dx$.

With this in mind, the differential equation (6) sometimes will be written in the differential form $dy = F(x,y)dx$ or in an algebraically equivalent form. For example, the differential equation

$$y' = \frac{3x^2}{x^3 + 1}(y + 1)$$

can be written in the form

$$dy = \left[\frac{3x^2}{x^3 + 1}(y + 1)\right]dx \quad \text{or} \quad y' - \frac{3x^2}{x^3 + 1}y = \frac{3x^2}{x^3 + 1}.$$

There are several types of first-order ordinary differential equations whose solutions can be found explicitly or implicitly by integrations. Of all tractable types of first-order ordinary differential equations, two deserve special attention: differential equations with *variables separable,* that is, equations that can be put into the form

$$y' = \frac{P(x)}{Q(y)} \quad \text{or} \quad P(x)dx = Q(y)dy,$$

and *linear equations,* that is, equations that can be put into the form

$$y' + a(x)y = b(x).$$

Both appear frequently in applications, and many other types of differential equations are reducible to one or the other of these types by means of a simple transformation.

MATHEMATICAL MODELS 1.1.1

Differential equations appear frequently in mathematical models that attempt to describe real-life situations. Many natural laws and hypotheses can be translated via mathematical language into equations involving derivatives. For example, derivatives appear in physics as velocities and accelerations, in geometry as slopes, in biology as rates of growth of populations, in psychology as rates of learning, in chemistry as reaction rates, in economics as rates of change of the cost of living, and in finance as rates of growth of investments.

It is the case with many mathematical models that in order to obtain a differential equation that describes a real-life problem, we usually assume that the actual situation is governed by very simple laws—which is to say that we often make idealistic assumptions. Once the model is constructed in the form of a differential equation, the next step is to solve the differential equation and utilize the solution to make predictions concerning the behavior of the real problem. In case these predictions are not in reasonable agreement with reality, the scientist must reconsider the assumptions that led to the model and attempt to construct a model closer to reality.

First-order ordinary differential equations are very useful in applications. Let the function $y = y(x)$ represent an unknown quantity that we want to study. We know from calculus that the first derivative $y' = dy/dx$ represents the rate of change of y per unit change in x. If this rate of change is known (say, by

EXERCISES

In Exercises 1 through 8, answer true or false.

1. $y = e^x + 3e^{-x}$ is a solution of the differential equation $y'' - y = 0$.

2. $y = 5 \sin x + 2 \cos x$ is a solution of the differential equation $y'' + y = 0$.

3. $y = \sin 2x$ is a solution of the differential equation $y'' - 4y = 0$.

4. $y = \cos 2x$ is a solution of the differential equation $y'' + 4y = 0$.

5. $y = \left(\frac{1}{2x}\right) e^{x^2}$ is a solution of the differential equation $xy' + y = xe^{x^2}$.

6. $y = e^x$ is a solution of the differential equation $y' + y = 0$.

7. $y = e^{-x}$ is a solution of the differential equation $y' - y = 0$.

8. $y = 2 \ln x + 4$ is a solution of the differential equation $x^2y'' - xy' + y = 2 \ln x$.

In Exercises 9 through 14, a differential equation and a function are listed. Show that the function is a solution of the differential equation. If more than one function is listed, show that both functions are solutions of the differential equation.

9. $y'' + y' = 2;\ 2x,\ 2x - 3$

10. $y'' - (\tan x)y' - \frac{\tan x}{x} y = \frac{1}{x^2} y^3;\ x \sec x$

11. $y''' - 5y'' + 6y' = 0;\ e^{3x}$

12. $y''' - 5y'' + 6y' = 0;\ e^{2x}, 1$

13. $m\ddot{s} = \frac{1}{2} gt^2;\ \frac{g}{24\,m} t^4 \left(\ddot{s} = \frac{d^2s}{dt^2}\right)$

14. $y'' - 2y' + y = \frac{1}{x}(y - y');\ e^x \ln x$

15. Show that the functions $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$ are solutions of the ordinary differential equation $y'' + 2y' + y = 0$.

16. Show that the function $y(x) = c_1e^{-x} + c_2xe^{-x} + 1$ is a solution of the ordinary differential equation $y'' + 2y' + y = 1$ for any values of the constants c_1 and c_2.

17. Show that the functions $y_1(x) \equiv 0$ and $y_2(x) = x^2/4, x \geq 0$, are solutions of the ordinary differential equation

$$y' = y^{1/2}.$$

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18. Show that the function

$$y(x) = \begin{cases} 0 & \text{for } x \le c \\ \dfrac{(x-c)^2}{4} & \text{for } x > c, \end{cases}$$

for any real number c, is a solution of the differential equation

$$y' = y^{1/2}.$$

(*Warning:* Don't forget to show that the solution is differentiable everywhere and in particular at $x = c$.)

19. Show that the function

$$x(t) = \frac{1}{k} + \frac{kx_0 - 1}{k} e^{-k(t-t_0)},$$

where x_0 and t_0 are constants, is a solution of Eq. (11). Show also that this solution passes through the point (t_0, x_0); that is, $x(t_0) = x_0$.

20. Show that $N(t) = ce^{kt}$ for any constant c is a solution of Eq. (7). What is the meaning of the constant c?

21. Find the differential equation of the orthogonal trajectories of the family of straight lines $y = cx$. Using your geometric intuition, guess the solution of the resulting differential equation.

In Exercises 22 through 28, answer true or false.

22. $y = xe^x$ is a solution of the differential equation $y' - 2y = e^x(1 - x)$.

23. $y = x^2$ is a solution of the differential equation $y''' = 0$.

24. $y = x + 1$ is a solution of the differential equation $yy' - y^2 = x^2$.

25. $y = \sin^2 x$ is a solution of the differential equation $y'' + y = \cos^2 x$.

26. $y = -1 + e^{-x}$ is a solution of the differential equation $y'' = y'(y' + y)$.

27. $y = -\sqrt{\dfrac{4 - x^3}{3x}}$ is a solution of the differential equation $y' = \dfrac{x^2 + y^2}{2xy}$.

28. $y = -\sqrt{\dfrac{4 - x^3}{3x}}$ is a solution of the differential equation $y' = -\dfrac{x^2 + y^2}{2xy}$.

29. Verify that each member of the one-parameter family of curves

$$x^2 + y^2 = 2cx$$

cuts every member of the one-parameter family of curves

$$x^2 + y^2 = 2ky$$

at right angles and vice versa. (*Hint:* Show that slopes are negative reciprocals.)