Answers – Practice Problems for Test 3 - MTH 141 Fall 2003 12/01

1. a. $f'(x) = 6x^2 + 6x - 72 = 6(x^2 + x - 12) = 6(x+4)(x-3)$. Hence from $f'(x) = 0$ we have that the critical points are $x = -4$, $x = 3$.

   b. $f''(x) = 12x + 6$; then,
   $f''(-4) = -42 \Rightarrow$ local max at $x = -4$; $f''(3) = 42 \Rightarrow$ local min at $x = 3$.

   c. $f''(x) = 12x + 6$, hence the only possible inflection point is $x = -\frac{1}{2}$.
   Check: $f''(-1) = -6 < 0$, $f''(0) = 6 > 0$. We conclude that $x = -\frac{1}{2}$ is an inflection point since there is a sign change of f'' at $x = -\frac{1}{2}$

2. a. $g'(x) = 12x^3 + 24x^2 - 12x - 24 = 12(x^3 + 2x^2 - x - 2)$. Now the equation $g'(x) = 0$ has the three solutions $x = -2$, $x = -1$ and $x = 1$. This can be seen from factoring $g'(x)$, or from some calculator work. The critical points are $x = -2$, $x = -1$, $x = 1$.

   b. $g''(x) = 36x^2 + 48x - 12$. Then,
   $g''(-2) = 36 \Rightarrow$ local min at $x = -2$.
   $g''(-1) = -24 \Rightarrow$ local max at $x = -1$.
   $g''(1) = 72 \Rightarrow$ local min at $x = 1$.

   c. $g''(x) = 36x^2 + 48x - 12$. By solving $g''(x) = 0$ we get the possible inflection points: $x = \frac{1}{3}(-2 - \sqrt{7}) \approx -1.548$, and $x = \frac{1}{3}(-2 + \sqrt{7}) \approx 0.2152$.
   Check: $g''(-2) = 36 > 0$, $g''(0) = -12 < 0$, $g''(1) = 72 > 0$. We conclude that both are inflection points.

3. a. $h'(t) = -e^{-t^2} - te^{-t^2}(-2t) = (-1 + 2t^2)e^{-t^2}$. Critical points: $t = -\frac{1}{\sqrt{2}}$, $t = \frac{1}{\sqrt{2}}$.

   b. $h''(t) = -t(4t^2 - 6)e^{-t^2}$. Then,
   $h''(-\frac{1}{\sqrt{2}}) = -2\sqrt{\frac{2}{e}} \Rightarrow$ local max at $t = -\frac{1}{\sqrt{2}}$.
   $h''(\frac{1}{\sqrt{2}}) = 2\sqrt{\frac{2}{e}} \Rightarrow$ local min at $t = \frac{1}{\sqrt{2}}$.

   c. $h''(t) = -t(4t^2 - 6)e^{-t^2}$.
   Possible inflection pts: $t = 0$, $t = -\sqrt{\frac{3}{2}} \approx -1.225$, $t = \sqrt{\frac{3}{2}} \approx 1.225$
   The following sign changes of h'' prove that the points we found are inflection points: $h''(-2) = \frac{20}{e^4} > 0$, $h''(-1) = -\frac{2}{e} < 0$, $h''(1) = \frac{2}{e} > 0$, $h''(2) = -\frac{20}{e^4} < 0$.

4. a. $R'(t) = \frac{4\,(1+t)}{2+2\,t+t^2}$, hence the only critical point is $t = -1$.

   b. The second derivative is $R''(t) = \frac{-4\,t\,(2+t)}{(2+2\,t+t^2)^2}$. Then,
   $R''(-1) = 4 \Rightarrow$ local min at $t = -1$.

   c. From the equation $R''(t) = 0$ we have the possible inflection points: $t = -2$, $t = 0$.
   The following sign changes of h'' prove that the points we found are inflection points: $f''(-3) = -\frac{12}{25} < 0$, $f''(-1) = 4 > 0$. $f''(1) = -\frac{12}{25} < 0$.

5. Note: in our text the endpoints of the interval are considered critical points (no derivative there), and also as possible points of local max or min. The following table reflects this.

Kind of point	Point or Points
Critical point	E, G, A, I
Inflection point	D, F
Local min	G
Local max	E , I , A
Global min	G
Global max	A

6. $f(\frac{1}{3}) = 1 \Rightarrow \frac{a}{3}e^{b/3} = 1$, and $f'(\frac{1}{3}) = 0 \Rightarrow a(1 + \frac{b}{3})e^{b/3} = 0$, which implies that either $a = 0$ or $b = -3$. The $a = 0$ case gives the trivial solution $f(x) = 0$ for all x, which contradicts $f(\frac{1}{3}) = 1$, so we discard this case. Then we must have $b = -3$, which combined with $\frac{a}{3}e^{b/3} = 1$ gives $a = 3e$. That we have a local maximum at $x = 1/3$ is clear from the first derivative test, or from a plot of $f(x)$.

7. x-intercepts: 0, $-2a$. y-intercepts: 0. Critical points: $x = -a$. $f''(-a) = 2 > 0$, hence $f(x)$ has a local minimum at $x = -a$. Since $f(x)$ is a parabola, the local minimum is also a global minimum. The figure corresponds to $a > 0$. If a is increased, the nonzero x-intercept moves away (left) from the origin, and the minimum moves "down".

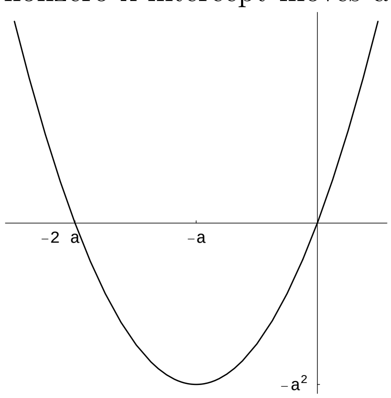
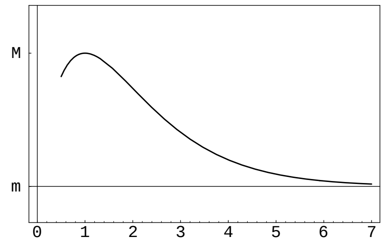


Figure for problem 7 when $a > 0$.

8. (a) $\{f(0), f(3), f(4)\} = \{0, -135, -112\} \Rightarrow x_{min} = 3$, $x_{max} = 0$,
   (b) $\{f(-5), f(-4), f(3), f(4)\} = \{185, 208, -135, -112\} \Rightarrow x_{min} = 3$, $x_{max} = -4$,
   (c) $\{f(-2), f(2)\} = \{140, -116\} \Rightarrow x_{min} = 2$, $x_{max} = -2$,
   (d) $\{f(4), f(10)\} = \{-112, 1580\} \Rightarrow x_{min} = 4$, $x_{max} = 10$.

9. (a) $\{f(-2), f(-\frac{1}{\sqrt{2}}), f(0)\} \approx \{0.036, 0.4288, 0.\} \Rightarrow x_{min} = 0$, $x_{max} = -\frac{1}{\sqrt{2}}$,
   (b) $\{f(-2), f(-\frac{1}{\sqrt{2}}), f(\frac{1}{\sqrt{2}}), f(2)\} \approx \{0.036, 0.4288, -0.4288, -0.036\}$
   $\Rightarrow x_{min} = \frac{1}{\sqrt{2}}$, $x_{max} = -\frac{1}{\sqrt{2}}$,
   (c) $\{f(-0.5), f(0.5)\} = \{0.3894, -0.3894\} \Rightarrow x_{min} = 0.5$, $x_{max} = -0.5$,
   (d) $\{f(-3), f(-1)\} = \{0.00037, 0.3678\} \Rightarrow x_{min} = -3$, $x_{max} = -1$,

10. Let us call m and M the best bounds. A plot of $f(x)$ suggests the answer $M = f(1) \approx 0.3678$ and $m = 0$.



Auxiliary plot for question 10.

We verify this as follows: $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, hence $x = 1$ is the only critical point. Since $f'(0.5) \approx 0.303$ and $f'(2.0) \approx -0.1353$, we see that there is a local maximum at $x = 1$ (which is also a global maximum, since there is only one critical point). Therefore, $M = f(1) \approx 0.3678$. To get m, we investigate behavior of $f(x)$ near the endpoints of the interval. Observe that $f(0.5) \approx 0.3032$ and that $\lim_{x\to\infty} f(x) = 0$. Therefore, $m = 0$.

11. The function to be optimized is cost "C". If the length and width of the rectangle are called x and y, and if the side with fancy fence has length x, the total cost of the fence is

$$C = 7.5x + 5y + 5x + 5y = 12.5x + 10y$$

We must reduce C to a single variable. For this we use the fact that the total area is 2880:

$$xy = 2880 \quad \Rightarrow \quad y = \frac{2880}{x}$$

Substituting this value of y into C we have the formula

$$C = 12.5x + \frac{28800}{x}, \quad \text{valid for} \quad 0 < x < \infty$$

We shall need

$$C' = 12.5 - \frac{28800}{x^2} \quad \text{and} \quad C'' = \frac{57600}{x^3}$$

Solving for x in $C' = 0$ gives $x = \pm 48$, but the negative answer is discarded. So $x = 48$ is the only critical point, and since $C''(48) \approx 0.52$, we have a local minimum at $x = 48$. Since this is the only critical point, C also has a global minimum there. Substituting $x = 48$ into $y = \frac{2880}{x}$, we get $y = 60$. Conclusion: the optimal dimensions of the rectangle are $x = 48$, $y = 60$, and the optimal cost of the fence is $C(48) = 1200$.

12. If the rectangle has sides x and y, the function to be minimized is the diagonal $d = \sqrt{x^2 + y^2}$. To reduce d to a sigle variable, use the fact $xy = A$, where A is a given constant which we assume known. Since $y = A/x$, we have

$$d = \sqrt{x^2 + \left(\frac{A}{x}\right)^2} = \sqrt{x^2 + \frac{A^2}{x^2}}, \quad \text{where} \quad 0 < x < \infty$$

Now $d' = \frac{2\,x - \frac{2\,A^2}{x^3}}{2\,\sqrt{x^2 + \frac{A^2}{x^2}}}$, hence $d' = 0 \Rightarrow x = \pm\sqrt{A}$. The negative answer is discarded, and we are left with a single critical point, namely $x = \sqrt{A}$. Now

$$d''(x) = \frac{2\,\left(A^4 + 3\,A^2\,x^4\right)}{x^6\,\left(x^2 + \frac{A^2}{x^2}\right)^{\frac{3}{2}}} \quad \Rightarrow \quad d''(\sqrt{A}) = \frac{2\sqrt{2}}{\sqrt{A}} > 0$$

so $x = \sqrt{A}$ gives a local minimum of d which is also a global minimum. Hence the dimensions of the rectangle of area A that has the shortest diagonals are $x = \sqrt{A}$ and $y = A/x = \sqrt{A}$.

13. See page 205 of the text.

14.
$$(\tanh t)' = \left(\frac{\sinh t}{\cosh t}\right)' = \frac{\cosh t \cosh t - \sinh t \sinh t}{\cosh^2 t} = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t}$$

15. A: CFT, B: IVT, C: MVT, D: EVT

16. a)Upper estimate = Right Sum = 146; Lower estimate = Left Sum = 122;

    b) From $\Delta t\,|v(b) - v(a)|$ = *difference between upper and lower estimates*, and the requirement that the difference be less than 0.1, we have that $\Delta t = \frac{difference}{|v(b)-v(a)|} < \frac{0.1}{290-170} \approx 0.0008333$.

    c) from $\Delta t = \frac{b-a}{n}$, we have $0.0008333 > \Delta t = \frac{0.8-0.2}{n}$, hence $n > \frac{0.8-0.2}{0.0008333} \approx 720$.

17. a)Upper estimate = Left Sum = 111; Lower estimate = Right Sum = 0.57;

    b) From $\Delta t\,|Q'(b) - Q'(a)|$ = *difference between upper and lower estimates* and from the requirement that the difference be less than 0.0001, we have that $\Delta t = \frac{difference}{|Q'(b)-Q'(a)|} < \frac{0.0001}{|0.19-0.01|} \approx 0.0005556$

    c) from $\Delta t = \frac{b-a}{n}$, we have $0.0005556 > \Delta t = \frac{15-3}{n}$, hence $n > \frac{15-3}{0.0005556} \approx 21600$.

18. (a) 6, (b) 12, (c) 3, (d) 12, (e) 6, (f) 3.

19. a) Left sum $= -5.6$, Right sum $= 5.0$, $\int_0^{10} f(t)dt \approx \frac{1}{2}(Left + Right) = -0.3$.

    b) The average value of $f(t)$ on $0 \le t \le 10$ is $\frac{1}{10-0}\int_0^{10} f(t)dt \approx \frac{1}{10} \cdot (-0.3) = -0.03$.

20. a) Left sum $= \frac{4375}{27} \approx 162.037$, Right sum $= \frac{4375}{27} \approx 162.037$.

    b) The answers are the same due to the symmetry of the graph of the function $f(x) = -x^2 + 10x$ with respect to the vertical line $x = 5$, and to the fact that $f(x)$ increases for $x \le 5$, and that $f(x)$ decreases for $x \ge 5$. The Left and Right sums are not necessarily equal to the integral, even though Left=Right. Indeed, a calculator computation (using `fnInt` on a Texas Instruments calculator, for example) gives that the integral ≈ 166.66.

21. Left sum $= \left(\sin(0) + \sin(\frac{25}{9}) + \sin(\frac{100}{9})\right)\frac{5}{3} \approx -1.06241$

    Right sum $= \left(\sin(\frac{25}{9}) + \sin(\frac{100}{9}) + \sin(25)\right)\frac{5}{3} \approx -1.28307$

22. Left sum $= \left(e^{-(-1)^2} + e^{-(-0.8)^2} + e^{-(-0.6)^2} + e^{-(-0.4)^2} + e^{-(-0.2)^2}\right)0.2 \approx 0.681156$,

    Right sum $= \left(e^{-(-0.8)^2} + e^{-(-0.6)^2} + e^{-(-0.4)^2} + e^{-(-0.2)^2} + e^{-(0)^2}\right)0.2 \approx 0.80758$.

23. (a) The furthest the car gets ahead of the truck is at $t \approx 1.5$ hrs. The separation then is approximately "3 rectangles" = 30 miles. However, at the end of the 7 hours the truck is ahead by approximately "12 rectangles" = 120 miles !

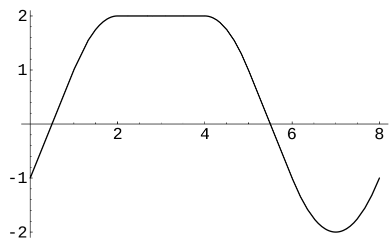
    b) Yes, they do meet at approximately $t = 3.5$ hours.

c) At time $t = 6$ hrs the car comes to a stop, while the truck keeps going at ≈ 34 mph. At time $t = 7$, the car is going in direction opposite to that of the truck, with velocity = -10 mph, while the truck moves at 45 mph.

24.

x	0	1	2	3	4	5	6	7	8
$f(x)$	1	0	1	3	4	4	4	3	1

25.



26. (a) $\int_0^5 3f(t) + 2g(t)\,dt = 3\int_0^5 f(t)dt + 2\int_0^5 g(t)dt = 3 \cdot 23 + 2 \cdot 15 = 99$

(b) Cannot be done: we need the value of $\int_2^5 g(t)dt$ to answer (b).

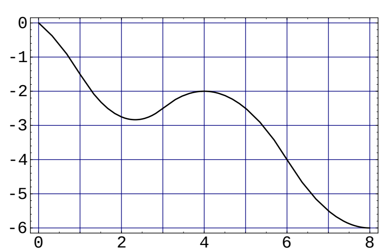
(c) $\int_0^5 g(t)dt = \int_0^3 g(t)dt + \int_3^5 g(t)dt$, hence $15 = \int_0^3 g(t)dt + 22$, so $\int_0^3 g(t)dt = -7$.

(d) $\int_5^0 g(t)dt = -\int_0^5 g(t)dt = -15$.

27.

x	0	1	2	3	4	5	6	7	8
$f(x)$	1	-1	-2	-2	-2	-1	1	2	1

28.



Critical points: $x = 2.3$ and $x = 4$; Inflection points: $x = 1$, $x = 3$ and $x = 6$.