Lecture 7

Counting Zeros and the Open Mapping Theorem

- Counting Zeros of an analytic function
- Counting zeros II
- Counting solutions to $f(z) = b$
- Number of solutions to $f(z) = b$ is “stable”
- Open Mapping Theorem
- Meromorphic functions and the argument principle

Counting Zeros of an analytic function

Theorem 16 *Let $f(z)$ be analytic in an open set G and such that*

$$f(z) = (z - a_k) \cdots (z - a_n) g(z) =: p(z) g(z)$$

where $g(z) \neq 0$ in G. If $\gamma \sim 0$ in G and $a_k \notin \{\gamma\}$ for $k = 1, \ldots, n$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} \operatorname{wind}(\gamma; a_k)$$

Proof. We have,

$$\begin{aligned} \tfrac{f'(z)}{f(z)} &= \left(\Sigma_{k=1}^{n} \times_{j \neq k} (z - a_j)\right) g(z) + p(z) g'(z) \\ &= p(z) g(z) \left(\Sigma_{k=1}^{n} \tfrac{1}{z - a_k}\right) + p(z) g'(z) \\ &= f(z) \left(\Sigma_{k=1}^{n} \tfrac{1}{z - a_k}\right) + p(z) g'(z) \end{aligned}$$

Since γ is contractible to a point in G ($\gamma \sim 0$) and g'/g is analytic on G, we may apply Cauchy's theorem to obtain

$$\int_\gamma \frac{g'(z)}{g(z)} dz = 0$$

Hence,

$$\begin{aligned} \textstyle\int_\gamma \tfrac{f'(z)}{f(z)} dz &= \textstyle\int_\gamma \Sigma_{k=1}^{n} \tfrac{n}{z - a_k} dz + \int_\gamma \tfrac{g'(z)}{g(z)} dz \\ &= \Sigma_{k=1}^{n} 2\pi i \operatorname{wind}(\gamma; a_k) + 0 \end{aligned}$$

Counting zeros II

Consider a closed curve γ enclosing a connected, simply connected open set Δ. If γ is oriented positively (counterclockwise), then we have,

$$\mathsf{wind}(\gamma, a) = \begin{cases} 1 & \text{if } a \in \Delta \\ \\ 0 & \text{if } a \notin \Delta \end{cases}$$

Then,

$$\begin{aligned} \tfrac{1}{2\pi i} \textstyle\int_\gamma \tfrac{f'(z)}{f(z)} dz &= \Sigma_{k=1}^{n} \mathrm{wind}(\gamma; a_k) \\ &= \text{Number of zeros of } f \text{ in } \Delta \end{aligned}$$

Counting solutions to $f(z) = \alpha$

Corollary 8 *Let f and γ be as in the preceding theorem, except that $a_1, \ldots, a_n$ are the points in G that satisfy the equation $f(z) = \alpha$. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}dz = \sum_{k=1}^{n} \text{wind}(\gamma; a_k)$$

In particular, if γ is a simple closed curve oriented positively and that encloses a connected, simply connected set, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}dz = \text{No. of solutions of } f(z) = \alpha \text{ in } \Delta$$

Number of solutions to $f(z) = b$ is "stable"

Let γ be a closed curve that bounds a region G that is simply connected, and let $f(z)$ be analytic on a set that includes as subsets G and $\{\gamma\}$. Then $\sigma = fo\gamma$ is a curve in the complex plane, and we have,

$$\begin{aligned} \text{wind}(\sigma, b) &= \tfrac{1}{2\pi i} \int_\sigma \tfrac{dw}{w-b} = \tfrac{1}{2\pi i} \int_\gamma \tfrac{f'(z)}{f(z)-b} dz \\ &= \text{No. solns. in } G \text{ of } f(z) = b \end{aligned}$$

The path of σ divides up the complex plane into connected components of $\mathbf{C} - \{\sigma\}$. Each component consists of points b so that the number of solutions in G of $f(z) = b$ is constant throughout the component. See figure.

Open Mapping Theorem

Theorem 17 *Let G be a region and suppose that $f(z)$ is a nonconstant analytic function on G. Then, for any open set $H \subset G$, the set $f(H)$ is open.*

Proof. We first prove that for any disk $B(a, \epsilon) \subset G$, the set $f(B(a, \epsilon))$ contains an open disk $B(f(a), \delta)$. Once this is proved, the conclusion of the theorem follows from the fact that the union of such $B(f(a), \delta)$ disks is precisely $f(H)$.
We have that if γ is a circle with radius ϵ traversed counterclockwise once , then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-f(a)}dz = n \geq 1$$

Now $f(a)$ belongs to an open connected component U of $C - \{f(\gamma)\}$. We may choose $\delta > 0$ such that $B(f(a), \delta) \subset U$. Then,

$$z \in \{\gamma\} \text{ and } w \in B(f(a), \delta) \Rightarrow f(z) - w \neq 0$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-w}dz = n \geq 1 \quad \text{for all } w \in B(f(a), \delta)$$

That is, $f(z) = w$ has $n \geq 1$ solutions for each $w \in B(f(a), \delta)$. $\square$

Meromorphic functions and the Argument Principle

Definition 14 *If G is an open set in* **C** *and if $f(z)$ is a function analytic on G except for poles, we say that $f(z)$ is meromorphic on G.*

Theorem 18 *Let $f(z)$ be meromorphic on G open with zeros $z_1, \dots, z_n$ and poles $p_1, \dots, p_m$, counted according to multiplicity. Let γ be a curve in G such that $\gamma \sim 0$ and $z_\ell \notin \{\gamma\}$, $p_\ell \notin \{\gamma\}$. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{\ell=1}^{n} \textit{wind}(\gamma; z_\ell) - \sum_{\ell=1}^{m} \textit{wind}(\gamma; p_\ell)$$

Proof. If $f(z)$ has a zero of order m at $z = a$, then we may write $f(z) = (z-a)^m g(z)$ where $g(z) \neq a$ and $g(z)$ is analytic in a neighborhood of $z = a$. Then,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)} \tag{1}$$

and $\frac{g'(z)}{g(z)}$ is analytic near $z = a$.
Now if $f(z)$ has a pole of order m at $z = a$, then $f(z) = (z-a)^{-m} g(z)$ where $g(z) \neq a$ and $g(z)$ is analytic in a neighborhood of $z = a$. Then,

$$\frac{f'(z)}{f(z)} = -\frac{m}{z-a} + \frac{g'(z)}{g(z)} \tag{2}$$

and $\frac{g'(z)}{g(z)}$ is analytic near $z = a$.

In the general case where $f(z)$ has poles and zeros, we have that by applying (1) and (2) that

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k} - \sum_{k=1}^{m} \frac{1}{z - p_k} + \frac{g'(z)}{g(z)}$$

Note that $g(z)$ is analytic and does not vanish on G, so $\frac{g'(z)}{g(z)}$ is also analytic. Integrate along γ both sides of (**??**) and apply Cauchy's theorem to get

$$\int_{\gamma} \frac{g'(z)}{g(z)} dz = 0$$

and to obtain the conclusion of the theorem. $\square$