§ 4.8 Conditional Expectations

We have already seen how to calculate the expectation of a r.v. given its distribution.

The conditional distributions introduced in § 3.7 lead to conditional expectations.

Defn If X is a discrete r.v. and $f(x|y)$ is the value of the conditional pdf of X given $Y=y$ at x, the conditional expectation of $u(X)$ given $Y=y$ is

$$E(u(X)|y) = \sum_x u(x)\cdot f(x|y) = \sum_x u(x)\cdot \frac{f(x,y)}{h(y)}$$

$(h(y) \neq 0)$.

If X is a cts. r.v. and $f(x|y)$ is the value of the conditional pdf of X given $Y=y$ at x, the <u>conditional expectation</u> of $u(X)$ given $Y=y$ is

$$E(u(X)|y) = \int_{-\infty}^{\infty} u(x) \cdot f(x|y)\, dx = \int_{-\infty}^{\infty} u(x) \cdot \frac{f(x,y)}{h(y)}\, dx$$

$(h(y) \neq 0)$.

If we let $u(X) = X$ in the above defn, we obtain the <u>conditional mean</u> of the r.v. X given $Y=y$, which we denote by

$$\mu_{X|y} = E(X|y).$$

We also have the conditional variance of X given $Y = y$, $\sigma^2_{X|y}$ where

$$\sigma^2_{X|y} = E\left((X - \mu_{X|y})^2 \mid y\right)$$

$$= E(X^2 \mid y) - \mu^2_{X|y}.$$

Ex. The tablets yet again!

Find the conditional mean of X given $Y = 1$.

Sol^n. Earlier we found

$$f(0|1) = \frac{4}{7}, \quad f(1|1) = \frac{3}{7}, \quad f(2|1) = 0.$$

Thus.

$$E(X|1) = 0 \cdot f(0|1) + 1 \cdot f(1|1) + 2 \cdot f(2|1)$$

$$= 0 \cdot \frac{4}{7} + 1 \cdot \frac{3}{7} + 2 \cdot 0 = \frac{3}{7}.$$

This gives the expected number of aspirins given that one of the two tablets selected was a sedative.

<u>Ex</u>. If X, Y are cts with joint pdf

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

find the conditional mean and the conditional variance of X given $Y = \frac{1}{2}$.

<u>Soln</u> We already saw earlier (§ 3.7) that for these r.v.'s the conditional density of X given $Y = y$ is

$$f(x|y) = \begin{cases} \frac{2x+4y}{1+4y}, & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Thus

$$f(x \mid \tfrac{1}{2}) = \begin{cases} \frac{2}{3}(x+1), & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

and $\mu_{X \mid \frac{1}{2}}$ is given by

$$E(X \mid \tfrac{1}{2}) = \int_{-\infty}^{\infty} x\, f(x \mid \tfrac{1}{2})\, dx$$

$$= \int_0^1 \frac{2}{3} x (x+1)\, dx$$

$$= \frac{5}{9}.$$

Next we find

$$E(X^2 \mid \tfrac{1}{2}) = \int_0^1 \frac{2}{3} x^2 (x+1)\, dx$$

$$= \frac{7}{18}.$$

Finally

$$\sigma^2_{X|\frac{1}{2}} = E(X^2|\tfrac{1}{2}) - \left(E(X|\tfrac{1}{2})\right)^2$$

$$= \frac{7}{18} - \left(\frac{5}{9}\right)^2$$

$$= \frac{13}{162}.$$