We may also consider the case of two cts. r.v.'s.

<u>Defn</u>. A bivariate fn with values $f(x,y)$ defined on $\mathbb{R}^2$ is called a <u>joint probability density function</u> of the cts. r.v.'s X and Y iff

$$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy$$

for any region A in the xy-plane.

Analagous to Thm 3.5 for one cts r.v., we have

<u>Theorem 3.8</u> A bivariate fn can serve as a joint pdf of a pair of cts. r.v.'s X, Y if its values $f(x,y)$ satisfy

1. $f(x,y) \geq 0, \quad -\infty < x,y < \infty$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

Ex. Given the joint pdf

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x), & 0<x<1,\ 0<y<2 \\ 0, & \text{elsewhere} \end{cases}$$

of two r.v.'s X, Y, find $P((X,Y)\in A)$ where A is the region

$$A = \{(x,y): 0<x<\tfrac{1}{2},\ 1<y<2\}.$$

Soln.

$$P((X,Y)\in A) = P(0<X<\tfrac{1}{2},\ 1<Y<2)$$

$$= \int_1^2 \int_0^{\frac{1}{2}} \frac{3}{5}x(y+x)\,dx\,dy$$

$$= \int_1^2 \left[\frac{3x^2y}{10} + \frac{3x^3}{15}\right]_0^{\frac{1}{2}} dy$$

$$= \int_1^2 \left(\frac{3y}{40} + \frac{1}{40}\right)dy = \left[\frac{3y^2}{80} + \frac{y}{40}\right]_1^2 = \frac{11}{80}.$$

We can also define a joint CDF for two cts. r.v.'s.

<u>Defn.</u> If X, Y are cts. r.v.'s, the fn given by

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt, \quad -\infty < x, y < \infty$$

where $f(x,y)$ is the value of a joint pdf of X, Y is called the <u>joint cumulative distribution function</u> of X and Y.

Similarly to the univariate case.

<u>Thm</u> If $F(x,y)$ is the value of the joint CDF of two cts r.v.'s X & Y at (x,y), then

a) $\lim\limits_{\substack{x \to -\infty \\ y \to -\infty}} F(x,y) = 0$, b) $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F(x,y) = 1$

c) if $a < b$ and $c < d$, then $F(a,c) \le F(b,d)$.

Similarly to $f(x) = \dfrac{dF(x)}{dx}$ for one cts. r.v.,

for two cts r.v.'s we have

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

whenever these partial derivatives exist.

So as before

$$\text{Joint CDF} \xrightarrow{\frac{\partial^2}{\partial x \partial y}} \text{Joint pdf.}$$

As in §3.4, we let $f(x,y) = 0$ whenever the above relationship doesn't hold.

Ex. If the joint pdf of X & Y is given by

$$f(x,y) = \begin{cases} x+y , & 0<x,y<1 \\ 0 , & \text{elsewhere} \end{cases}$$

find the joint CDF of these two r.v.'s.

Soln. If either $x<0$ or $y<0$, then $F(x,y)=0$.

e.g.. If $x<0$

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\,ds\,dt$$

$$= \int_{-\infty}^{y} 0\,dt \qquad \text{as } f(s,t)=0 \text{ as } s \le x < 0$$

$$=0$$

(Region 0)

If $0 < x < 1$, $0 < y < 1$ (Region I),

$$F(x,y) = \int_0^y \int_0^x (s+t)\,ds\,dt = \frac{1}{2} xy(x+y).$$

If $1 < x$, $0 < y < 1$ (Region II),

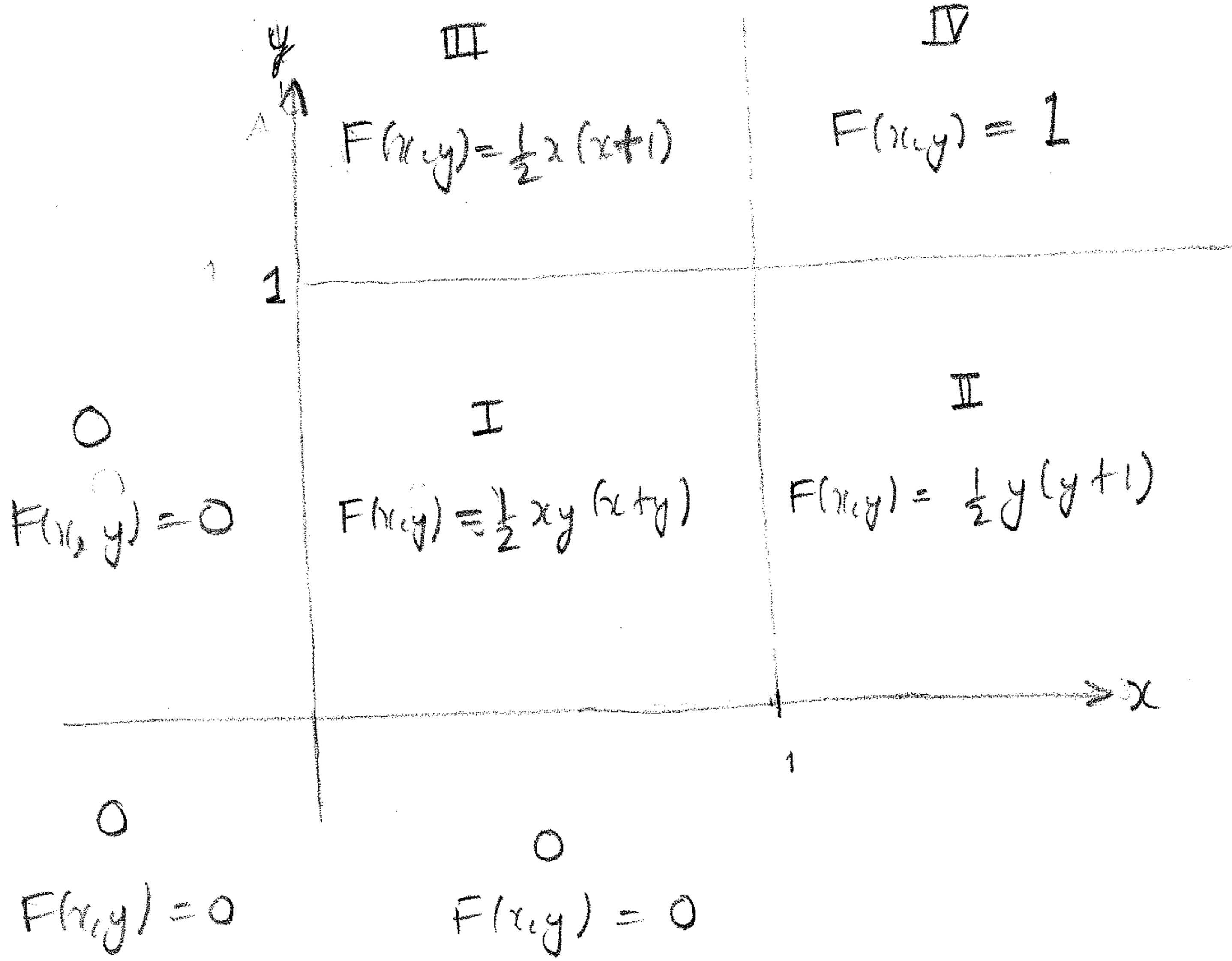
$$F(x,y) = \int_0^y \int_0^1 (s+t)\,ds\,dt = \frac{1}{2} y(y+1)$$

If $0 < x < 1$, $1 < y$ (Region III),

$$F(x,y) = \int_0^1 \int_0^x (s+t)\,ds\,dt = \frac{1}{2} x(x+1)$$

If $1 < x$, $1 < y$ (Region IV),

$$F(x,y) = \int_0^1 \int_0^1 (s+t)\,ds\,dt = 1.$$



So

$$F(x,y) = \begin{cases} 0, & x \le 0,\ y \le 0 \\ \frac{1}{2}xy(x+y), & 0<x<1,\ 0<y<1 \\ \frac{1}{2}y(y+1), & 1<x,\ 0<y<1 \\ \frac{1}{2}x(x+1), & 0<x<1,\ 1<y \\ 1, & 1\le x,\ 1 \le y. \end{cases}$$

N.b. Since F is cts. everywhere, it doesn't matter just which region we say the boundary lines (e.g. $x=1$) belong to.

Ex. Find the joint pdf of the two r.v.'s X and Y whose joint CDF is given by

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) \ , & x, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Also use the joint pdf to find

$P(1 < X < 3, \ 1 < Y < 2)$.

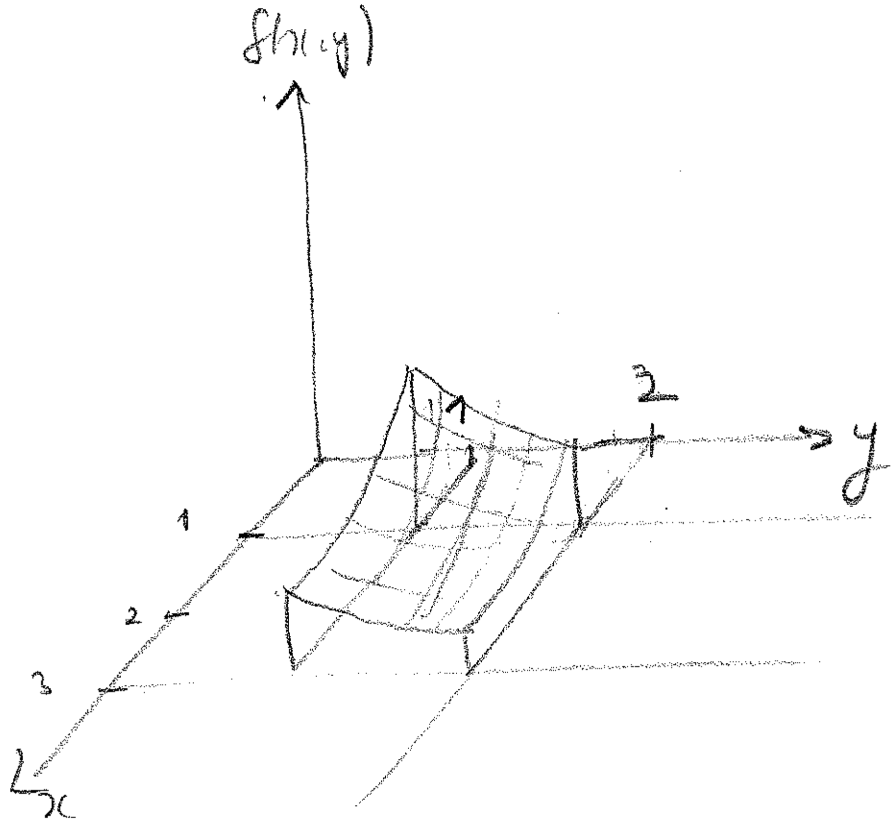
Soln

Partial differentiation gives

$$\frac{\partial^2 F}{\partial x \partial y} = e^{-(x+y)} \quad \text{if} \quad x, y > 0$$

and $$\frac{\partial^2 F}{\partial x \partial y} = 0 \quad \text{if} \quad x < 0 \text{ or } y < 0.$$

We can then set $f(x,y) = 0$ on the remaining pts $(x = 0$ or $y = 0)$ to get e.g. For $P(1 < X < 3, 1 < Y < 2)$, the picture looks like.



All the defns in this section can be generalized to the multivariate case where we have n r.v.'s

The joint pdf of n discrete r.v.'s is given by

$$f(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

for each n-tuple $(x_1, \ldots, x_n)$ within the range of the r.v's.

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x>0,\ y>0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, by integration

$P(1<X<3,\ 1<Y<2)$

$$= \int_1^2 \int_1^3 e^{-(x+y)}\,dx\,dy = (e^{-1}-e^{-3})(e^{-1}-e^{-2})$$

$$= e^{-2}-e^{-3}-e^{-4}+e^{-5}$$

$$\approx 0.074$$

In terms of multivariable calculus, we can think of $f(x,y)$ as a surface and probability corresponds to finding the volume under the surface on a certain region

The joint CDF is given by

$$F(x_1, x_2, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$$

for $-\infty < x_1, \ldots, x_n < \infty$.

Ex. If the joint pdf of 3 discrete r.v's is given by

$$f(x, y, z) = \frac{(x+y)z}{63}, \quad \begin{array}{l} x = 1, 2 \\ y = 1, 2, 3 \\ z = 1, 2 \end{array},$$

find $P(X=2, Y+Z \leq 3)$.

Soln.

$$P(X=2, Y+Z \leq 3) = f(2,1,1) + f(2,1,2) + f(2,2,1)$$

$$= \frac{3}{63} + \frac{6}{63} + \frac{4}{63}$$

$$= \frac{13}{63}.$$

In the cts case, probs. are again obtained by integrating a pdf and

$$P((X_1, \ldots, X_n) \in A) = \underset{A}{\iint \cdots \int} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

for any region $A \subset \mathbb{R}^n$.

The joint CDF is given by

$$F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

$$= \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(t_1, t_2, \ldots, t_n)\, dt_1\, dt_2 \cdots dt_n,$$

$$-\infty < x_1, \ldots, x_n < \infty.$$

Also partial diff gives

$$f(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} F(x_1, x_2, \ldots, x_n)$$

whenever these partial derivatives exist.

Ex. If the trivariate prob. density of X_1, X_2, X_3 is given by

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2) e^{-x_3}, & 0 < x_1 < 1,\ 0 < x_2 < 1,\ 0 < x_3 \\ 0 & \text{otherwise.} \end{cases},$$

find $P((X_1, X_2, X_3) \in A)$ where

$$A = \left\{ (x_1, x_2, x_3) ;\ 0 < x_1 < \tfrac{1}{2},\ \tfrac{1}{2} < x_2 < 1,\ x_3 < 1 \right\}.$$

Sol^n.

$$P((X_1, X_2, X_3) \in A) = P\left(0 < X_1 < \tfrac{1}{2}, \tfrac{1}{2} < X_2 < 1,\ X_3 < 1\right)$$

$$= \int_0^1 \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (x_1 + x_2) e^{-x_3} \, dx_1 \, dx_2 \, dx_3$$

$$= \int_0^1 \int_{\frac{1}{2}}^1 \left(\frac{1}{8} + \frac{x_2}{2} \right) e^{-x_3} \, dx_2 \, dx_3$$

$$= \int_0^1 \frac{1}{4} e^{-x_3} \, dx_3 = \frac{1}{4}(1 - e^{-1}) \approx 0.158.$$