2.8 Bayes' Theorem

An example where the outcome of an experiment depends on an intermediate stage with 2 alternatives.

Ex. The completion of a construction job may be delayed by a strike.

Prob. of a strike is 0.6

Prob job will be finished on time if there is no strike is 0.85

Prob. job will be finished on time if there is a strike is 0.35.

What is the prob. the job will be finished on time?

Let $A = \{\text{job is finished on time}\}$

$B = \{\text{there is a strike}\}$

Want $P(A)$.

Then A is a disjoint union

$$A = (A \cap B) \cup (A \cap B^c)$$

and so

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$= P(B) \cdot P(A \mid B) + P(B^c) \cdot P(A \mid B^c)$$

$$= 0.6 \times 0.35 + (1 - 0.6) \times 0.85$$

$$= 0.55.$$

We can also consider situations where there are more alternatives for the intermediate stage.

<u>Defn.</u> A countable collection $B_1, B_2, \ldots, B_i, \ldots$ of events is called a <u>partition</u> of a sample space S if they are pairwise disjoint and

$$\bigcup_i B_i = S.$$

Theorem 2.12 (Law of Total Probability or Rule of Elimination)

If the events $B_1, B_2, \ldots, B_i, \ldots$ constitute a partition of the sample space S and $P(B_i) \neq 0$ for each i, then for any event $A \subset S$,

$$P(A) = \sum_i P(B_i) \cdot P(A \mid B_i)$$

Pf is similar to argument in last example.

Ex. A firm rents cars from 3 agencies:
60% from agency 1, 30% from agency 2, 10% from agency 3. If 9% of the cars from agency 1 need a tune-up and 20% of those from agency 2 and 6% from agency 3, what is the chance that a rental car delivered to the firm needs a tune-up?

Let $A = \{\text{car needs a tune up}\}$

$B_1 = \{\text{car comes from agency } 1\}$

$B_2 = \{\text{car comes from agency } 2\}$

$B_3 = \{\text{car comes from agency } 3\}$

Then $P(B_1) = 0.6$, $P(B_2) = 0.3$, $P(B_3) = 0.1$

$P(A|B_1) = 0.09$, $P(A|B_2) = 0.2$, $P(A|B_3) = 0.06$.

By Thm 2.12

$$P(A) = 0.6 \times 0.09 + 0.3 \times 0.2 + 0.1 \times 0.06$$
$$= 0.12.$$

Suppose instead we asked: if a rental car needs a tune-up, what is the prob. it comes from agency 2?
(ie $P(B_2|A)$ rather than $P(A|B_2)$)
To answer this, we need Bayes' thm.

Thm 2.13 Bayes' Theorem

If $B_1, B_2, \ldots, B_i, \ldots$ constitute a partition of a sample space S and $P(B_i) \neq 0$ for each i, then for any event $A \subset S$ with $P(A) \neq 0$, we have

$$P(B_i|A) = \frac{P(B_i) \cdot P(A|B_i)}{\sum_j P(B_j) \cdot P(A|B_j)} \quad \text{for each } i.$$

PF. $P(B_i) \cdot P(A|B_i) = P(A \cap B_i)$ by Thm 2.9

while $\sum_j P(B_j) \cdot P(A|B_j) = P(A)$ by Thm 2.12

Thus

$$\frac{P(B_i) \cdot P(A \mid B_i)}{\sum_j P(B_j) \cdot P(A \mid B_j)} = \frac{P(A \cap B_i)}{P(A)} = P(B_i \mid A)$$

by defn of conditional prob.

<u>Note</u> we have $P(B_i \mid A)$ on the lhs and $P(A \mid B_i)$, $P(A \mid B_j)$ on the rhs, so the order of conditioning is reversed.

In the previous example, the prob. a car is from agency 2 given that it needs a tune-up is given by

$$P(B_2 \mid A) = \frac{P(B_2) \cdot P(A \mid B_2)}{\sum_{j=1}^{3} P(B_j) \cdot P(A \mid B_j)}$$

$$= \frac{0.3 \times 0.2}{0.6 \times 0.09 + 0.3 \times 0.2 + 0.1 \times 0.06}$$

$$= \frac{0.06}{0.12} = 0.5$$

A famous 'paradox'!

Ex. In a certain population, 1% have a particular disease, while the other 99% does not have this disease. A test for this disease

a) detects disease 98% of the time when disease is present

b) detects disease .5% of the time when disease is absent (false positive).

What is the prob. a person is sick given the test is positive?

Let $S = \{\text{person is sick}\}$
$T = \{\text{test is positive}\}$

Then $P(S) = .01$, $P(T|S) = .98$, $P(T|S^c) = .005$.

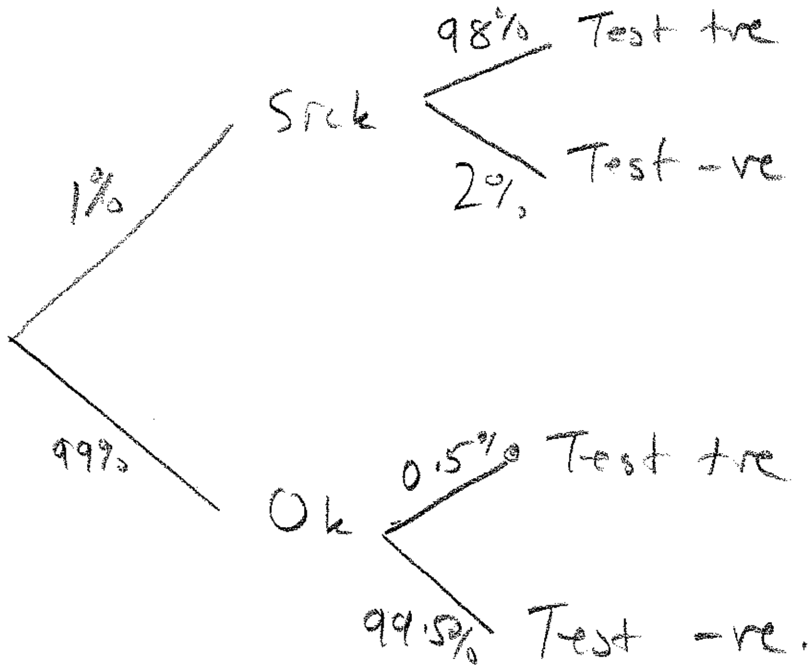
Want $P(S|T)$.

By Bayes

$$P(S|T) = \frac{P(S)P(T|S)}{P(S)P(T|S) + P(S^c)P(T|S^c)}$$

$$= \frac{.01 \times .98}{.01 \times .98 + .99 \times .005} \approx \frac{2}{3} \quad !$$

Can make a tree diagram



Ex. Another Paradox - The Goats and the Ferrari, or Let's Make a Deal.

In the 1970's game show Let's Make a Deal, the host, Monty Hall, would show contestants three closed doors. Behind one door was a valuable prize (e.g. a Ferrari) while behind the other two doors were objects of no value (e.g. goats).

The contestant was asked to pick one of the three doors. The host then opened one of the other two doors to reveal a goat behind it and asked the contestant if he would like to change his initial choice to the one remaining door.

In order to maximise the chances of winning, should the contestant switch or not, or does it indeed make any difference?

Soln To be definite, let's say the contestant picks door #1 and the host opens door #3.

Let $D_i = \{\text{the host opens door } \#i\}, \quad i=1,2,3$

$P_i = \{\text{the prize is behind door } \#i\}, \quad i=1,2,3$

In order to find if the contestant should switch, we need $P(P_2 \mid D_3)$.

By Bayes' Theorem (Thm 2.13),

$$P(P_2 \mid D_3) = \frac{P(P_2) \cdot P(D_3 \mid P_2)}{P(P_1)P(D_3 \mid P_1) + P(P_2)P(D_3 \mid P_2) + P(P_3)P(D_3 \mid P_3)}$$

All of these probs. are known and we get

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1 + \frac{1}{3} \times 0} = \frac{2}{3}.$$

Similarly $P(P_1 \mid D_3) = \frac{1}{3}$, so the contestant should switch.