Chapter 6 Special Probability Densities

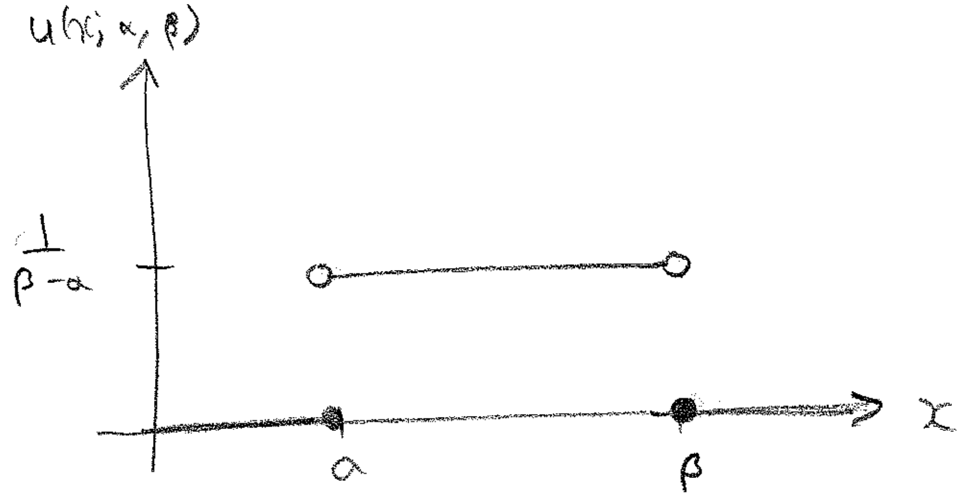
We continue our collection of commen pdf's this time for cts. r.v.'s.

§ 6.2 Uniform Distribution

The simplest density there is.

<u>Defn</u>. A r.v. has a uniform(α, β) distribution iff its pdf can be written

$$u(x; \alpha, \beta) = \begin{cases} \dfrac{1}{\beta - \alpha}, & \alpha < x < \beta \\ 0 & \text{elsewhere.} \end{cases}$$



<u>Thm 6.1</u> The mean & variance of the uniform (α, β) distr. are given by

$$\mu = \frac{\alpha+\beta}{2}, \quad \sigma^2 = \frac{1}{12}(\beta-\alpha)^2.$$

Pf. Easy ex.

§ 6.3 The Exponential Distribution

This distribution is used to model the waiting time for some rare event (e.g. road accident, radioactive decay, bus arriving) to occur.

<u>Defn</u> A r.v. has an exponential(λ) distribution for some $\lambda > 0$ iff its pdf can be written

$$g(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Ex. Check $\int_{-\infty}^{\infty} g(x;\lambda) = 1$.

<u>Fact</u> the exp(λ) distn. has mean $\frac{1}{\lambda}$ and variance λ^{-2}.

Pf. Ex - Integration by parts!

<u>Fact</u>. The MGF of an exp(λ) r.v. is

$$M_X(t) = \frac{1}{1 - t/\lambda}, \qquad t < \lambda.$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{xt} g(x;\lambda)\,dx$$

$$= \int_0^{\infty} e^{xt} \frac{e^{-\lambda x}}{\lambda}\,dx$$

$$= \lambda \int_0^{x} e^{-(\lambda - t)x}\,dx$$

$$= \lambda \cdot \frac{1}{\lambda - t} = \frac{1}{1 - t/\lambda} \qquad \text{(provided } t < \lambda\text{)}$$

The value $\frac{1}{\lambda}$ is often called the arrival time of the $\exp(\lambda)$. r.v.

In terms of waiting times, it is the average time one has to wait before an event occurs.

Ex. At exit 22 on I-95, one has to wait on average $\frac{1}{2}$ hr to see a car doing at least 20mph over the speed limit. Use an exponential r.v. to calculate the prob. that one will have to wait less than 15 min to see a car doing more than 20mph over the limit.

Soln. If we measure time in hours, then

$$\frac{1}{\lambda} = \text{arrival time} = \frac{1}{2}.$$

Thus $\lambda = 2$ and the prob. we are waiting < 15 min ($\frac{1}{4}$ hr) is given by

$$\int_0^{\frac{1}{4}} 2e^{-2x}\,dx$$

$$= \left[-e^{-2x}\right]_0^{\frac{1}{4}}$$

$$= 1 - e^{-\frac{1}{2}} \approx .393.$$

The exponential distr. can be thought of as a cts version of the geometric distr. where we measure time ctsly instead of in discrete intervals.

The exp. & Poisson distrs are related by the Poisson process which models the occurrences of rare events.
In the Poisson process:

- the number of events occurring in a given time interval is a Poisson r.v.
- the intervals (waiting times) between events are exponential r.v's.