§ 4.6 Product Moments

Defn The rth and sth product moment about the origin of the r.v.'s X, Y denoted $\mu'_{r,s}$ is the expectation of $X^r \cdot Y^s$. If X, Y are discrete, we have

$$\mu'_{r,s} = E(X^r Y^s) = \sum_x \sum_y x^r y^s f(x,y)$$

$r = 0, 1, 2, \ldots$
$s = 0, 1, 2, \ldots$

and if X, Y are cts.

$$\mu'_{r,s} = E(X^r Y^s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^r y^s \cdot f(x,y)\, dx\, dy$$

$r = 0, 1, 2, \ldots$
$s = 0, 1, 2, \ldots$

As with moments for one r.v., we can also define product moments about the means.

Defn Let X, Y be r.v.s with means μ_X, μ_Y resp. The rth and sth product moment about the means of X, Y denoted by $\mu_{r,s}$ is the expected value of $(X-\mu_X)^r(Y-\mu_Y)^s$. If X, Y are discrete, then

$$\mu_{r,s} = E\left((X-\mu_X)^r(Y-\mu_Y)^s\right)$$
$$= \sum_x \sum_y (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)$$

$r = 0, 1, 2, \ldots$
$s = 0, 1, 2, \ldots$

and if X, Y are cts, then

$$\mu_{r,s} = E\left((X-\mu_X)^r(Y-\mu_Y)^s\right)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)\,dx\,dy$$

$r = 0, 1, 2, \ldots$
$s = 0, 1, 2, \ldots$

Note that as with one r.v., one needs to consider if these sums and integrals converge.

Note also that

$$\mu_X = E(X) = E(X^1 Y^0) = \mu'_{1,0}$$

$$\mu_Y = E(Y) = E(X^0 Y^1) = \mu'_{0,1}$$

Of special importance, is $\mu_{1,1}$ as this is indicative of any relationship (correlation) between X and Y.

<u>Defn</u> $\mu_{1,1}$ is called the <u>covariance</u> of X and Y and it is denoted by σ_{XY}, $\mathrm{cov}(X,Y)$, or $C(X,Y)$.

A useful result for calculating covariances in practice.

<u>Theorem 4.11</u> $\sigma_{XY} = \mu'_{1,1} - \mu_X \mu_Y$.

<u>Pf.</u> $\sigma_{XY} = E((X-\mu_X)(Y-\mu_Y))$ by defn.

$$= E(XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y)$$

$$= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y$$

$$= \mu'_{1,1} - \mu_Y\mu_X - \mu_X\mu_Y + \mu_X\mu_Y$$ by defn.

$$= \mu'_{1,1} - \mu_X\mu_Y. \quad \blacksquare$$

<u>Remark</u> note the similarity with

$$\text{var}(X) = \sigma^2 = \mu_2' - \mu^2 \quad (\text{Thm } 4.6)$$

Indeed if $Y = X$, then we get Thm 4.6 as a special case of Thm 4.11.

Ex The tablets (again)

		x: 0	1	2	$h(y)$
y	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{7}{12}$
	1	$\frac{2}{9}$	$\frac{1}{6}$		$\frac{7}{18}$
	2	$\frac{1}{36}$			$\frac{1}{36}$
	$g(x)$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{12}$	

Then

$$\mu_{1,1} = E(XY) = 0\cdot 0\cdot \tfrac{1}{6} + 0\cdot 1\cdot \tfrac{2}{9} + 0\cdot 2\cdot \tfrac{1}{36} + 1\cdot 0\cdot \tfrac{1}{3} + 1\cdot 1\cdot \tfrac{1}{6} + 2\cdot 0\cdot \tfrac{1}{12}$$

$$= \frac{1}{6}$$

$$\mu_X = E(X) = \sum_x x\,g(x) = 0\cdot\tfrac{5}{12} + 1\cdot\tfrac{1}{2} + 2\cdot\tfrac{1}{12} = \frac{2}{3}$$

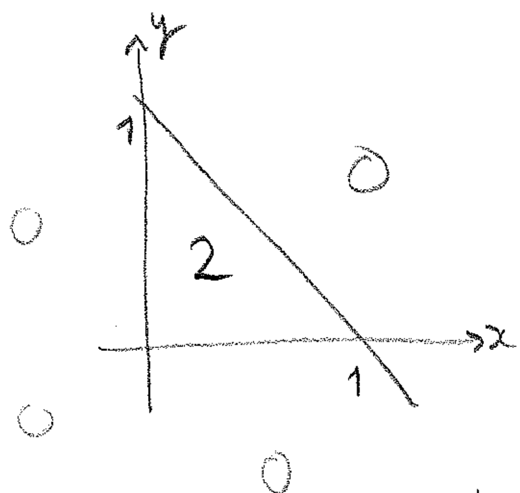
$$\mu_Y = E(Y) = \sum_y y\,h(y) = 0\cdot\tfrac{7}{12} + 1\cdot\tfrac{7}{18} + 2\cdot\tfrac{1}{36} = \frac{4}{9}$$

Then $\sigma_{XY} = \mu'_{1,1} - \mu_X\mu_Y = \frac{1}{6} - \frac{2}{3}\cdot\frac{4}{9} = -\frac{7}{54}$.

Note that $\sigma_{XY} < 0$ as we would expect (why?).

<u>Ex</u>. Find σ_{XY} for the pair of r.v.'s whose joint pdf is

$$f(x,y) = \begin{cases} 2, & x>0, y>0, \quad x+y<1 \\ 0 & \text{elsewhere.} \end{cases}$$



<u>Soln</u>. $\mu_X = \int_0^1 \int_0^{1-x} 2x \, dy \, dx = \frac{1}{3}$

$$\mu_Y = \int_0^1 \int_0^{1-x} 2y \, dy \, dx = \frac{1}{3}$$

Remark We could actually have figured this out without integration - why?

$$\mu'_{1,1} = \int_0^1 \int_0^{1-x} 2xy \, dx \, dy = \frac{1}{12}$$

Then

$$\sigma_{XY} = \mu'_{1,1} - \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{36}.$$

Q. What is the relationship between covariance and independence?

Thm 4.12 If X and Y are independent, then

$$E(XY) = E(X)E(Y) \quad \text{and}$$

$$\sigma_{XY} = 0.$$

Pf. (Discrete case).

Since X, Y are indep., by defn we have

$$f(x,y) = g(x)\,h(y)$$

where $g(x)$, $h(y)$ are the marginals in X, Y resp.

Then

$$E(XY) = \sum_x \sum_y xy\, f(x,y)$$

$$= \sum_x \sum_y xy\, g(x)\cdot h(y) \quad \text{by indep.}$$

$$= \left(\sum_x x\, g(x)\right)\left(\sum_y y\, h(y)\right)$$

$$= E(X)E(Y) \quad \text{as required.}$$

Also

$$\sigma_{XY} = \mu'_{1,1} - \mu_X \mu_Y = E(XY) - E(X)E(Y)$$

$$= E(X)E(Y) - E(X)E(Y) = 0.$$

So

$$X, Y \text{ indep.} \quad \Rightarrow \quad \operatorname{cov}(X, Y) = 0$$

However, the converse is <u>not</u> true in general.

Ex. Supp. X, Y have joint pdf and marginals as below

$y \backslash x$	-1	0	1	$h(y)$
-1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{3}$
0	0	0	0	0
1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$
$g(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

Then

$$\mu_X = (-1)\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = 0$$

$$\mu_Y = (-1)\cdot\frac{2}{3} + 0\cdot 0 + 1\cdot\frac{1}{3} = -\frac{1}{3}$$

and

$$\begin{aligned}\mu_{XY} &= (-1)(-1)\cdot\frac{1}{6} + 0(-1)\cdot\frac{1}{3} + 1(-1)\cdot\frac{1}{6} \\ &\quad + 0 + 0 + 0 \\ &\quad + (-1)\cdot 1\cdot\frac{1}{6} + 0 + 1\cdot 1\cdot\frac{1}{6} \\ &= 0\end{aligned}$$

Then $\sigma_{XY} = \mu'_{1,1} - \mu_X\mu_Y = 0 - 0\cdot(-\frac{1}{3})$
$= 0.$

However, X and Y are <u>not</u> indep.

Eg. $f(0,1) = 0$ but

$$g(0)h(1) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9} \neq 0.$$

Product moments can also be defined for three or more r.v.s. in the obvious way. The results are similar.

e.g. one has

Thm 4.13 If $X_1, X_2, \ldots, X_n$ are indep., then

$$E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2)\cdots E(X_n)$$

The proof is similar to that for $E(XY) = E(X)E(Y)$ (Thm 4.12) of which this is a generalization.