§ 1 Preliminaries - Combinatorics

In probability we will often need to work out the number of ways in which something can happen and for this we need combinatorics.

<u>Ex1.</u> A six-sided die is thrown 3 times. How many possible outcomes are there?

A. Each throw has 6 possibilities and the order matters. Total number of outcomes is

$$6 \times 6 \times 6 = 216$$

In general, if we perform k experiments in turn each of which has n possible outcomes, we have

$$n^k$$

possible outcomes.

Ex 2 Worker selects parts from 4 bins each of which contains 4, 3, 5, 4 parts resp. She can choose the parts in

$$4 \times 3 \times 5 \times 4 = 240$$

different ways.

In general, if we perform k experiments where experiment i has n_i outcomes for each $1 \leq i \leq k$, then we have

$$n_1 \times n_2 \times \cdots \times n_k$$

possible outcomes. (Theorem 1.2)

Permutations - Factorials

Ex 3. How many ways can I arrange the three letters a, b, c.

Of course, we could just list the possibilities

abc, acb, bac, bca, cab, cba.

which gives us 6.

Can also reason by saying the a can go in three positions (first, middle, last) the b can go in either of the remaining two free positions and the c must go in the last free position. This gives us

$$3 \times 2 = 6$$

possibilities again.

In general, the number of ways of arranging n objects in a list (where the order matters is)

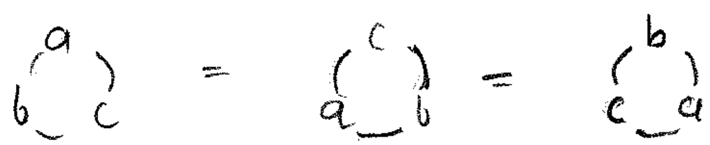
$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1. \qquad \text{(Thm 1.3)}$$

called the factorial of n. By convention

$$0! = \underline{1}. \qquad \text{(useful later).}$$

If we arrange our objects in a circle instead of a line and only care about the relative ordering (not the exact position)

e.g.



By symmetry, we can regard the first object as fixed (e.g. at 12 o' clock) while the other $n-1$ objects can be in any position.

This gives us

$$(n-1)!$$

arrangements. Called <u>circular permutations</u>. (Thm 1.5).

e.g. There are $4! = 24$ ways of seating 5 people round a circular table.

Another generalisation is if we want to make a list of k objects to be chosen from a set of n objects.

<u>Ex 4.</u> How many lists of 3 letters can we make from the letters a, b, c, d, e?

Have 5 choices for the first entry
4 ——— second (one letter used up)
3 ——— third (two letters used up)

Gives $5 \times 4 \times 3 = 60$ possibilities in total.

Can write this as

$$\frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

In general, the number of permutations of k objects chosen from a collection of n objects is

$$_nP_k = \frac{n!}{(n-k)!} \quad , \quad k = 0, \dots, n.$$

Thm 1.4

Binomial Coefficients - Combinations

Suppose again we draw k objects from a collection of n but this time we don't care about the order.

For a given choice of k objects, these can occur in any order which gives $k!$ possibilities.

Thus

$$\begin{gathered}\text{\# ways of choosing } k \text{ objects} \\ \text{from } n \text{ objects independent of order}\end{gathered}$$

$$= \frac{\begin{gathered}\text{\# ways of choosing } k \text{ objects} \\ \text{from } n \text{ objects with order}\end{gathered}}{\text{\# ways of ordering } k \text{ objects}}$$

$$= \frac{{}_nP_k}{k!} = \frac{\frac{n!}{(n-k)!}}{k!}$$

$$= \frac{n!}{k!\,(n-k)!} \qquad \text{(Thm 1.7)}.$$

This is usually written $\binom{n}{k}$ and it is called a <u>binomial</u> <u>coefficient</u>. Name comes from the binomial theorem which states

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k, \quad n \in \mathbb{N}.$$

(Thm 1.9).

Facts.

$$\binom{n}{k} = \binom{n}{n-k}, \quad n \in \mathbb{N},\ k = 0, \ldots, n$$

(Thm 1.10)

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}, \quad n \in \mathbb{N},\ k = 1, \ldots, n$$

Pascal's Theorem
(Thm 1.11)

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k} \qquad \begin{array}{l} m, n \geqslant 0 \\ 0 \leq k \leq n. \end{array}$$

n.b. Here we could have m or $n = 0$.

By convention

$$\binom{0}{k} := 0, \qquad k > 0. \qquad \text{(Thm 1.12)}.$$

<u>Ex.</u> The number of ways of choosing a poker hand of 5 cards from a deck of 52 cards is

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 2{,}598{,}960.$$

Multinomial Coefficients

Suppose we have 8 balls, 3 red, 3 green and 2 blue. How many different arrangements can we make.

Clearly swapping the red balls around doesn't change anything and there are $3! = 6$ ways to do this.

Similarly, the green & blue balls can be swapped in $3!$ & $2!$ ways respectively.

Thus the total number of permutations is

$$\frac{8!}{3!\,3!\,2!}$$

which is usually written as $\binom{8}{3,3,2}$.

In general, if we have n objects with k_1 of one type, k_2 of a second type, ---, k_j of a j^{th} type, then the number of permutations is

$$\binom{n}{k_1, k_2, \ldots, k_j} = \frac{n!}{k_1!\, k_2! \cdots k_j!},$$

$$n, j \in \mathbb{N}, \quad k_1, \ldots, k_j \geq 0$$
$$k_1 + k_2 + \cdots + k_j = n.$$

Note that if $j = 2$

$$\binom{n}{k_1, k_2} = \binom{n}{k_1, n-k_1} = \frac{n!}{k_1!\,(n-k_1)!} = \binom{n}{k_1},$$

so multinomial coefficients are a generalization of binomial coefficients.

<u>Ex</u>. Can you explain binomial coeffients in the same way as above?

We also have that for n distinct objects, the number of ways we can partition the collection into j subsets with k_1 going in the first subset, k_2 in the second, ..., and k_j in the jth is

$$\binom{n}{k_1, k_2, \ldots, k_j} \qquad \text{(Thm 1.8)}.$$

Multinomial coefficients arise in a similar way to binomial coefficients in the following generalization of the binomial theorem

$$(x_1 + x_2 + \cdots + x_j)^n = \sum_{\substack{0 \le k_1, \ldots, k_j \le n \\ k_1 + k_2 + \cdots + k_j = n}} \binom{n}{k_1, k_2, \ldots, k_j} x_1^{k_1} x_2^{k_2} \cdots x_j^{k_j}.$$

Ex. In how many ways can two paintings by Monet, three paintings by Renoir, and two paintings by Degas hang side by side on a museum wall if we do not distinguish between paintings by the same artist?

Here $n = 7$, $k_1 = 2$, $k_2 = 3$, $k_3 = 2$, so we have

$$\frac{7!}{2!\,3!\,2!} = 210$$

possible arrangements.

Ex. How many different committees of two chemists & one physicist can be formed from the 4 chemists & 3 physicists on the faculty of a small college?

There are $\binom{4}{2} = 6$ ways of choosing the chemists

and $\binom{3}{1} = 3$ ways of choosing the physicist.

This gives a total of $6 \times 3 = 18$ ways of forming the committee.