§ 4.7 Moments of Linear Combinations of Random Variables

<u>Thm 4.14</u> If $X_1, X_2, \ldots, X_n$ are r.v.'s and

$$Y = \sum_{i=1}^{n} a_i X_i$$

where $a_1, \ldots, a_n$ are consts, then

$$E(Y) = \sum_{i=1}^{n} a_i E(X_i)$$

and

$$\operatorname{var}(Y) = \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i) + 2 \sum_{i<j} a_i a_j \operatorname{cov}(X_i, X_j)$$

Pf. By Thm 4.5 with $g_i(X_1, \ldots, X_n) = X_i$, $1 \leq i \leq n$

$$E(Y) = E\left(\sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i E(X_i)$$

For var(Y), let us write μ_i for $E(X_i)$
$1 \leq i \leq n$.

Then

$$\text{var}(Y) = E(Y - E(Y))^2 \quad \text{by defn}$$

$$= E\left[\left(\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i E(X_i)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} a_i (X_i - \mu_i)\right)^2\right]$$

$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j (X_i - \mu_i)(X_j - \mu_j)\right]$$

$$= E\left[\sum_{i=1}^{n} a_i^2 (X_i - \mu_i)^2 + \sum\sum_{i \neq j} a_i a_j (X_i - \mu_i)(X_j - \mu_j)\right]$$

$$= E\left[\sum_{i=1}^{n} a_i^2 (X_i - \mu_i)^2 + 2 \sum\sum_{i < j} a_i a_j (X_i - \mu_i)(X_j - \mu_j)\right]$$

$$= \sum_{i=1}^{n} a_i^2 E((X_i - \mu_i)^2) + 2 \sum_{i<j}\sum a_i a_j E((X_i - \mu_i)(X_j - \mu_j))$$

$$= \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i) + 2 \sum_{i<j}\sum a_i a_j \operatorname{Cov}(X_i, X_j).$$

$\square$

This has the following important consequence. When $X_1, \ldots, X_n$ are (pairwise) independent, $\operatorname{Cov}(X_i, X_j) = 0$ if $i \neq j$ by Thm 4.12 and we get

<u>Corollary 4.3</u> If the r.v.'s $X_1, \ldots, X_n$ are (pairwise) independent and

$$Y = \sum_{i=1}^{n} a_i X_i,$$ then

$$\operatorname{var}(Y) = \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i).$$

Ex. Suppose X, Y, Z are s.t.

$$\mu_X = 2, \quad \mu_Y = -3, \quad \mu_Z = 4$$

$$\sigma_X^2 = 1, \quad \sigma_Y^2 = 5, \quad \sigma_Z^2 = 2$$

$$\text{cov}(X, Y) = -2, \quad \text{cov}(X, Z) = -1, \quad \text{cov}(Y, Z) = 1.$$

Find $E(W)$, $\text{var}(W)$ for $W = 3X - Y + 2Z$.

Soln. By Thm 4.14

$$E(W) = E(3X - Y + 2Z)$$

$$= 3E(X) - E(Y) + 2E(Z)$$

$$= 3 \cdot 2 - (-3) + 2 \cdot 4$$

$$= 17.$$

$$\text{var}(W) = 3^2 \text{var}(X) + (-1)^2 \text{var}(Y) + 2^2 \text{var}(Z)$$

$$+ 2 \cdot 3 \cdot (-1) \text{cov}(X, Y) + 2 \cdot 3 \cdot 2 \text{cov}(X, Z)$$

$$+ 2 \cdot (-1) \cdot 2 \text{cov}(Y, Z)$$

$$= 9 \cdot 1 + 5 + 4 \cdot 2 - 6(-2) + 12(-1) - 4 \cdot 1 = \underline{18}$$

The covariance of two linear combinations.

<u>Thm 4.15</u> If $X_1, X_2, \ldots, X_n$ are r.v.'s and

$$Y_1 = \sum_{i=1}^{n} a_i X_i, \quad Y_2 = \sum_{i=1}^{n} b_i X_i$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are consts, then

$$\operatorname{cov}(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \operatorname{var}(X_i)$$

$$+ \sum_{i<j}\sum (a_i b_j + a_j b_i) \cdot \operatorname{cov}(X_i, X_j).$$

Pf. Similar to that of Thm 4.14 - ex.

Again since $\text{cov}(X_i, X_j) = 0$ when X_i, X_j are ind (Thm 4.12), we have.

<u>Cor 4.4</u> If the r.v.'s $X_1, \ldots, X_n$ are indep,

$$Y_1 = \sum_{i=1}^{n} a_i X_i, \quad Y_2 = \sum_{i=1}^{n} b_i X_i, \quad \text{then}$$

$$\text{cov}(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \, \text{var}(X_i).$$

<u>Ex.</u> Supp X, Y, Z are s.t.

$$\mu_X = 3, \quad \mu_Y = 5, \quad \mu_Z = 2$$

$$\sigma_X^2 = 8, \quad \sigma_Y^2 = 12, \quad \sigma_Z^2 = 18$$

$$\text{cov}(X, Y) = 1, \quad \text{cov}(X, Z) = -3, \quad \text{cov}(Y, Z) = -3.$$

Find the covariance of

$$U = X + 4Y + 2Z \quad \text{and} \quad V = 3X - Y - Z.$$

Soln By Thm 4.15

$$\text{cov}(U, V) = \text{cov}(X + 4Y + 2Z, 3X - Y - Z)$$

$$= 1 \cdot 3 \text{ var } X + 4(-1) \text{ var } Y + 2(-1) \text{ var } Z$$

$$+ (1(-1) + 4 \cdot 3) \text{ cov}(X, Y)$$

$$+ (1(-1) + 2 \cdot 3) \text{ cov}(X, Z)$$

$$+ (4(-1) + 2(-1)) \text{ cov}(Y, Z)$$

$$= 3 \cdot 8 - 4 \cdot 12 - 2 \cdot 18$$

$$+ 11 \cdot 1 + 5(-3) - 6 \cdot 2$$

$$= -76$$