§ 4.3 Moments

Defn The <u>r-th moment about the origin</u> of a r.v. X, denoted by μ_r' is $E(X^r)$, the expected value of X^r.

In the discrete case

$$\mu_r' = E(X^r) = \sum_x x^r \cdot f(x)$$

and in the cts case

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x)\,dx .$$

n.b usually (nearly always), r is a non-negative integer ie $r = 0, 1, 2, \ldots$

N.b. the analogy with moments of inertia in physics, whence the name.

When $r=0$, $\mu_0' = E(X^0) = E(1) = 1$, by Cor 2 of Thm 4.2.

When $r=1$, $\mu_1' = E(X^1) = E(X)$, the expectation of X. This is also called the <u>mean</u> of the r.v. X.

<u>Defn</u>. μ_1' is called the mean of the distribution of X, or simply the mean of X and it is denoted by μ.

Of importance to us are the following.

<u>Defn</u>. The <u>rth moment about the mean</u> of a r.v. X, denoted by μ_r is the expected value of $(X-\mu)^r$.

In the discrete case

$$\mu_r = E((X-\mu)^r) = \sum_x (x-\mu)^r \cdot f(x)$$

and in the cts case

$$\mu_r = E((X-\mu)^r) = \int_{-\infty}^{\infty} (x-\mu)^r \cdot f(x)\,dx$$

Again, usually $r = 0, 1, 2, \ldots$

Note that $\mu_0 = 1$ and $\mu_1 = 0$
for any r.v. for which μ exists.
μ_2 is of special importance.

<u>Defn</u>. μ_2 is called the <u>variance</u> of the distribution of X, or simply the <u>variance</u> of X and it is denoted by σ^2, $var(X)$, or $V(X)$; σ, the positive square root of the variance is called the <u>standard deviation</u>

μ_2 (or σ) measures how closely the values of a r.v. are clustered around the mean.

μ_3 measures how asymmetrical about the mean the distribution of X is.

<u>Thm 4.6</u> $\sigma^2 = \mu_2' - \mu^2$

Pf. $\sigma^2 = E((X-\mu)^2)$

$= E(X^2 - 2\mu X + \mu^2)$

$= E(X^2) - 2\mu E(X) + E(\mu^2)$

$= E(X^2) - 2\mu \cdot \mu + \mu^2$

$= E(X^2) - \mu^2$

$= \mu_2' - \mu^2$. $\blacksquare$

One often uses Thm 4.6 to calculate the variance of a given r.v.

Ex. Use Thm 4.6 to calculate the variance of X, representing the number of points rolled with a balanced die.

Soln. $\mu = E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6}$

$= \frac{7}{2}$

$\mu_2' = E(X^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \cdots + 6^2 \cdot \frac{1}{6}$

$= \frac{91}{6}$

Then by Thm 4.6

$\sigma^2 = \mu_2' - \mu^2 = \frac{91}{6} - \frac{49}{4}$

$= \frac{35}{12}$.

Ex. Recall the example of a cts r.v. X in the last section whose pdf was

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

We already showed $\mu = E(X) = \frac{\ln 4}{\pi} \approx .4413$.

$$\mu_2' = E(X^2) = \frac{4}{\pi} \int_0^1 \frac{x^2}{1+x^2}\, dx$$

$$= \frac{4}{\pi} \int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{4}{\pi} \left(\int_0^1 1\, dx - \int_0^1 \frac{1}{1+x^2}\, dx \right)$$

$$= \frac{4}{\pi}\left(1 - \left[\arctan x\right]_0^1\right)$$

$$= \frac{4}{\pi}\left(1 - \left(\frac{\pi}{4} - 0\right)\right) = \frac{4}{\pi} - 1.$$

Thus.

$$\sigma^2 = \mu_2' - \mu^2 = \frac{4}{\pi} - 1 - \left(\frac{\ln 4}{\pi}\right)^2$$

$$\approx .0785$$

and

$$\sigma \approx \sqrt{.0785} \approx .2802$$

Another useful result.

<u>Thm 4.7</u> If X has variance σ^2, then

$$\mathrm{Var}(aX+b) = a^2\sigma^2$$

Pf. Ex.