§ 7.8 The Inverse of a Matrix
Gauss - Jordan Elimination

In this section, we consider only square matrices

For an $n \times n$ matrix $A = (a_{ij})$, the inverse A^{-1} is another $n \times n$ matrix for which

$$A A^{-1} = A^{-1} A = I_n \quad (n \times n \text{ identity matrix}).$$

Not all matrices A have an inverse A^{-1}!

If A^{-1} exists, we say A is invertible or non-singular. Otherwise we say A is non-invertible or singular.

Note that if A has an inverse, then it is unique. To see this, suppose B & C are both inverses for A. Then

$$B = I_n B = (CA) B \qquad \text{as } CA = I_n$$

$$= C(AB) \qquad \text{associativity of matrix multiplication}$$

$$= C I_n \qquad \text{as } AB = I_n$$

$$= C$$

So $B = C$. and there can only be one inverse.

Thm 1 Existence of the Inverse

The inverse A^{-1} of an $n \times n$ matrix A exists iff $\operatorname{rank} A = n$, thus (by Thm 4 in the last section) iff $\det A \neq 0$.

Hence if A is nonsingular, $\operatorname{rank} A = n$ and A is singular $\operatorname{rank} A < n$.

($\Rightarrow$) Suppose A^{-1} exists, let $a_1, \ldots, a_n$ be the column vectors of A and suppose $x_1, \ldots, x_n$ are scalars s.t.

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = \underline{0}.$$

Can write this in matrix form as

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underline{0}, \quad \text{or}$$

$$Ax = \underline{0}$$

Now multiply both sides on the left by A^{-1}

$$A^{-1}(A\,x) = A^{-1}\underline{0}$$

$$(A^{-1}A)\,x = A^{-1}\underline{0}$$

$$I_n\,x = \underline{0}$$

$$x = \underline{0}.$$

What we've shown is that if

$$x_1 a_1 + x_2 a_2 + \quad + x_n a_n = \underline{0}, \text{ then}$$

we must have $x_1 = x_2 = \cdots = x_n = 0$,

i.e. the columns of A are lin ind. and so rank $A = n$.

($\Leftarrow$) Suppose now that rank $A = n$.

Recall now that this means that the reduced & row echelon forms of A will have n pivots and by Gaussian elimination, for any $\underline{b} \in \mathbb{R}^n$

$$A x = b$$

has a <u>unique</u> soln.

Hence we can find x_1 for which $Ax_1 = e_1$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, the first standard basis vector for $\mathbb{R}^n$.

Similarly we can find x_2 for which $Ax_2 = e_2$

where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

etc.

Continuing in this way, we get vectors $x_1, x_2, \ldots, x_n$

with

$$Ax_i = e_i, \quad 1 \le i \le n \quad \left(\begin{array}{l} e_i \text{ is the} \\ i\text{th standard} \\ \text{basis vector} \\ \text{for } \mathbb{R}^n \end{array} \right)$$

Thus

$$A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix} = I_n$$

$$= I_n.$$

If we then let X be the $n \times n$ matrix of columns

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix},$$

we have just shown that $AX = I_n$.

Now if A has rank n, so does A^T (see §7.4) and if we do the same procedure above for A^T instead of A, then we can find another $n \times n$ matrix Y with

$$A^T Y = I_n$$

Now let $W = Y^T$. Then

$$A^T Y = I_n, \quad \text{so}$$

$$(A^T Y)^T = I_n^T = I_n$$

$$Y^T (A^T)^T = I_n \qquad \text{(remember } (AB)^T = B^T A^T\text{)}$$

$$Y^T A = I_n \qquad \text{(remember } (A^T)^T = A\text{)}.$$

$$WA = I_n \qquad \text{as } W := Y^T$$

So now we have one matrix X with $AX = I_n$ and another matrix W with $WA = I_n$.

Then

$$W = WI_n = W(AX) = (WA)X = I_n X$$

Hence, we have a matrix X s.t.

$$AX = XA = I_n.$$

Thus A is invertible and $X = A^{-1}$.

Determination of the Inverse by the Gauss - Jordan Method

Recall how in the proof of the last thm we found a matrix X whose columns x_i we solns of the n linear systems

$$Ax_i = e_i, \qquad 1 \leq i \leq n.$$

We then found later that in fact $X = A^{-1}$.

In practice, we would solve these systems by working on the n augmented matrices

$$[A \,|\, e_i].$$

In fact, if A is invertible, rank $A = n$ as we have just seen and since there are n pivots, it turns out that the reduced echelon form of A is just I_n and after row operations, we get

$$[I_n \,|\, x_i].$$

A less cumbersome way of doing this is to combine the information of these n augmented matrices into one 'double' augmented matrix

$$[A \mid I_n] \qquad \left(\text{recall } \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} = I_n \right)$$

and then do row operations on this until the lhs is I_n.

<u>Bottom Line</u> (What you need to remember).

If A is invertible, then one can find A^{-1} by row reduction of the double augmented matrix

$$[A \mid I_n]$$

until the left hand part is I_n. The right hand part is then A^{-1}, ie. one obtains

$$[I_n \mid A^{-1}]$$

Ex. Find the inverse of

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R2 + 3R1, \; R3 - R1$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R3 - R2$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Note. l.h. part now in row echelon form (corresponds to first part of Gaussian elimination).

$-R1, \; \frac{R2}{2}, \; \left(-\frac{1}{5}\right) R3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$R1 + 2R2, \quad R2 - \frac{7}{5}R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{13}{10} & -\frac{1}{5} & -\frac{7}{10} \\ 0 & 0 & 1 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$R1 + R2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 1 & 0 & -\frac{13}{10} & -\frac{1}{5} & -\frac{7}{10} \\ 0 & 0 & 1 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

So
$$A^{-1} = \begin{bmatrix} -7/10 & \frac{1}{5} & \frac{3}{10} \\ -\frac{13}{10} & -\frac{1}{5} & -\frac{7}{10} \\ \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

<u>Note</u>: It is rather easy to make mistakes in this procedure and it is a good idea to check your answer at the end by multiplying by multiplying by A, eg. check.

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}\begin{bmatrix} -\frac{7}{10} & \frac{1}{5} & \frac{3}{10} \\ -\frac{13}{10} & -\frac{1}{5} & -\frac{7}{10} \\ \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ie $AA^{-1} = I_3$ (Note: no need to also check $A^{-1}A = I_3$ as this is automatic).

Useful Formulae for Inverses

Thm 2 Inverse of a Matrix.

The inverse of an invertible $n \times n$ matrix $A = (a_{ij})$ is given by

$$(4) \quad A^{-1} = \frac{1}{\det A} (C_{ij})^T = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & \ddots & & \vdots \\ \vdots & & & \vdots \\ C_{1n} & \cdots & & C_{nn} \end{bmatrix}$$

where C_{ij} is the ij-th cofactor in $\det A$ (<u>NOTE</u> we are using the <u>transpose</u> $(C_{ij})^T$ and <u>not</u> (C_{ij}) itself).

In particular, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is 2×2, then

$$(4^*) \quad A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Pf. Try the easy 2×2 case. (4*)

$$\frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$$

$$= \frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{22}a_{12} - a_{12}a_{22} \\ -a_{21}a_{11} + a_{11}a_{21} & -a_{21}a_{12} + a_{11}a_{22}\end{bmatrix}$$

$$= \frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21}\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$$

A similar calculation shows

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} \cdot \frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} \quad \text{also}$$

and so we have proved the formula in this case.

General case. Let B be the matrix on the r.h.s. in (4). Want to show $BA = I_n$.

So first write

$$BA = G = (g_{ij})$$

and then show $G = I_n$.

Now, for $1 \leq i, j \leq n$

$$g_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$$

$$= \frac{1}{\det A} \sum_{k=1}^{n} C_{ki} a_{kj}$$

(recall, we have the transpose of the matrix of cofactors).

$$= \frac{1}{\det A} \sum_{k=1}^{n} a_{kj} C_{ki}$$

Now this is the determinant formed by cofactor expansion along column i of the matrix A' where

$$A' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & \vdots & & \vdots \\ \vdots & & & \vdots & & \vdots \\ a_{n1} & \cdots & & a_{nj} & & a_{nn} \end{bmatrix}$$

$\uparrow$

ith column

which we obtain from A by replacing the ith column of A with the jth column of A.

- If $i=j$, then we do nothing, so $A' = A$ and $g_{ii} = \dfrac{\det A'}{\det A} = \dfrac{\det A}{\det A} = 1$.
- If $i \neq j$, then the jth column of A appears <u>twice</u>, once as the ith column and once as the jth column. Recall from Thm 3 in the last section that if one

column of a matrix is proportional to another, then the det. is 0. Hence

$g_{ij} = 0$ if $i \neq j$.

Thus $g_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

i.e.

$$G = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} = I_n$$

Thus $BA = I_n$.

A similar argument (using cofactor expansion along rows rather than columns), shows that

$AB = I_n$.

Hence $B = A^{-1}$ as desired. $\blacksquare$

Ex.

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

Then $$A^{-1} = \frac{1}{3\times 4 - 1\times 2} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{3}{10} \end{bmatrix}$$

Ex.

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$ (same as before)

Soln. Find $\det A = -1 \times -7 - 1 \times 13 + 2 \times 8 = 10$
(cofactor exp. along row 1).

Also

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2$$

$$, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

$$C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2$$

Putting it all together

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{10} & \frac{1}{5} & \frac{3}{10} \\ -\frac{13}{10} & -\frac{1}{5} & \frac{7}{10} \\ \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

as before.

Fact Diagonal matrices $D = (d_{ij})$ with $d_{ij} = 0$ for $i \neq j$ are invertible iff $d_{ii} \neq 0$ for each $1 \leq i \leq n$.

Pf.

D is invertible iff $\det D \neq 0$ by Thm 1.

But $\det D = d_{11} d_{22} \cdots d_{nn}$ as D is diagonal (and in particular triangular - see § 7.7).

Hence

$$
\begin{gathered}
D \text{ is invertible} \\
\Updownarrow \\
d_{11} d_{22} \cdots d_{nn} \neq 0 \\
\Updownarrow \\
d_{11} \neq 0,\ d_{22} \neq 0, \ldots, d_{nn} \neq 0
\end{gathered}
$$

If

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & O \\ & & \ddots & \\ O & & & d_{nn} \end{bmatrix}$$

is invertible, then

$$D^{-1} = \begin{bmatrix} \frac{1}{d_{11}} & & & \\ & \frac{1}{d_{22}} & & O \\ & & \ddots & \\ O & & & \frac{1}{d_{nn}} \end{bmatrix}$$

and it is easy to check for this matrix that

$$D D^{-1} = D^{-1} D = I_n .$$

Ex. If

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \text{then}$$

$$D^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}.$$

Products of Invertible Matrices

Fact If A, B are invertible, then so is the product AB and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(note the similarity with the formula $(AB)^T = B^T A^T$ for transposes.)

Pf. Let $C = B^{-1}A^{-1}$

Then
$$\begin{aligned}(AB)C &= (AB)(B^{-1}A^{-1}) \\ &= A(BB^{-1})A^{-1} \\ &= A I_n A^{-1} \\ &= AA^{-1} \\ &= I_n.\end{aligned}$$

and
$$\begin{aligned}C(AB) &= (B^{-1}A^{-1})(AB) \\ &= B^{-1}(A^{-1}A)B \\ &= B^{-1} I_n B\end{aligned}$$

$$= B^{-1}B$$
$$= I_n.$$

Thus

$$(AB)C = C(AB) = I_n$$

and so $C = B^{-1}A^{-1}$ is the inverse of AB and in particular AB is invertible.

The same sort of argument also works for products of 3 or more invertible matrices.

e.g. If A, B, C are invertible, then so is ABC and

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

If $A_1, \ldots, A_m$ are invertible, then so is $A_1 A_2 \ldots A_m$ and

$$(A_1 A_2 \ldots A_m)^{-1} = A_m^{-1} \ldots A_2^{-1} A_1^{-1}.$$

Inverse of the Inverse

If A is invertible, then $(A^{-1})^{-1} = A$.

Pf.

$$
\begin{aligned}
(A^{-1})^{-1} &= (A^{-1})^{-1} I_n \\
&= (A^{-1})^{-1}(A^{-1}A) \\
&= \left((A^{-1})^{-1}A^{-1}\right)A \\
&= I_n A \\
&= A.
\end{aligned}
$$

Inverse of the Transpose

If A is invertible, then so is A^T and

$(A^T)^{-1} = (A^{-1})^T$ (inverse and transpose can be swapped).

Pf. Try it yourself!

Determinants of Matrix Products

Thm 4 For any $n \times n$ matrices A, B

$$\det(AB) = \det A \cdot \det B.$$

Remark This implies in particular that $\det(AB) = \det(BA)$ even though it is not true in general that $AB = BA$.

Pf. By row ops., can reduce A to its reduced echelon form which is a diagonal matrix D where

$$D = \begin{bmatrix} \underbrace{\begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}}_{r} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

, here $r = \operatorname{rank} D = \operatorname{rank} A$.

Now $\det A = c \det D$ for some (non-zero) constant D.

Subtle (Important) Points

1. Doing the same row ops. to the product AB reduces AB to DB.

e.g. $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ so $AB = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$

Suppose we add $3R1$ to $R2$ of A to get a new second row of a matrix $\tilde{A}$ where

$$\tilde{A} = \begin{bmatrix} 1 & 1 \\ 2+3\times 1 & 0+3\times 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2+3 & 0+3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$$

Then

$$\tilde{A}B = \begin{bmatrix} 1 & 1 \\ 2+3 & 0+3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2\times 1 + 3\times 1 + 0\times 2 + 3\times 2 & 2\times 3 + 3\times 3 + 0\times -1 + 3\times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ (2\times 1 + 0\times 2) + 3(1\times 1 + 1\times 2) & (2\times 3 + 0\times -1) + 3(1\times 3 + 1\times -1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2+3(3) & 6+3(2) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 11 & 12 \end{bmatrix}$$

which is the same matrix we get by adding 3(R1) of AB to R2 of AB to get R2 of a new matrix.

2. The constant c in $\det A = c \det D$ depends only on the row ops used (not on the matrix to which they are applied).

e.g. Multiplying the first row of a matrix C by 2 will double the det of C, regardless of what C actually is.

[illegible]

Combining 1 & 2 gives that if we carry out the same row op's to AB as we did to A to get D, we get DB and

$$\det(AB) = c \det(DB) \qquad (*)$$

Now

$$DB = \begin{bmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \cdots & & b_{2n} \\ \vdots & & & \vdots \\ b_{r1} & \cdots & & b_{rn} \\ 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}$$

Top r rows of DB are same as those of B.

Bottom $n-r$ rows of DB are zero.

Now if $r = \text{rank } A = n$, then $D = I_n$, so

$$DB = I_n B = B$$

and $\det(DB) = \det B = 1 \det B = \det D \det B$.

If $r = \text{rank } A < n$, then $\det D = 0$

while $\det DB = 0$ also as this as

$n - r \geq 1$ rows of zeroes.

Thus $\det(DB) = 0 = \det D \det B$

in this case also.

Now combine with ⊛ to get

$$\det(AB) = c \det(DB) = c \det D \det B$$

$$= (c \det D) \det B$$

$$= \det A \det B$$

as required. $\blacksquare$

Unusual Properties of Matrix Multiplication

Recall that for real numbers a, b, c we always have

1. $ab = ba$ – (commutative).

2. $ab = 0 \Rightarrow a = 0$ or $b = 0$ (or both)

3. $ab = ac,\ a \neq 0 \Rightarrow b = c.$
(cancellation)

None of these three properties hold for multiplication of matrices.

1. Already saw (§7.2) that it is not true in general that $AB = BA$.

2. $AB = 0$ does not generally imply that $A = 0$ or $B = 0$.

e.g.
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. $AB = AC$ does not generally imply that $B = C$.

Under what conditions might 2. & 3. be true?

<u>Thm 3</u> Cancellation Laws.

Let A, B, C be $n \times n$ matrices. Then:

a) If rank $A = n$ and $AB = AC$, then $B = C$;

b) If rank $A = n$, then $AB = 0 \Rightarrow B = 0$.
Hence if $AB = 0$, but $A \neq 0$, $B \neq 0$
then rank $A < n$ and rank $B < n$ also.

c) If A is singular then so are BA and AB.

Pf. a) If rank $A = n$, then A^{-1} exists by Thm 1.
Multiply both sides of
$$AB = AC$$
on the left by A^{-1} to get

$$A^{-1}(AB) = A^{-1}(AC)$$
$$(A^{-1}A)B = (A^{-1}A)C$$
$$I_n B = I_n C$$
$$\therefore B = C \quad \text{as required.}$$

b). If rank $A = n$, then again A^{-1} exists.

So

$$AB = 0$$

$$\Rightarrow \quad A^{-1}(AB) = 0$$

$$(A^{-1}A)B = 0$$

$$I_n B = 0$$

$$B = 0 \qquad \text{as required.}$$

c) Suppose A is singular and let $a_1, \ldots, a_n$ be the column vectors of A.

By thm 1, rank $A < n$, so $\{a_1, \ldots, a_n\}$ is lin dep which means we can find scalars $x_1, \ldots, x_n$ not all 0 s.t.

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0.$$

or

$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If we then let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then we have

$$A x = \underline{0}, \quad \text{so}$$

$$B(A \underline{x}) = 0$$

or $(BA)\underline{x} = 0$.

This implies that the cols. of BA are lin'l dep. and satisfy the same dependence relation as those of AB. Thus rank $BA < n$ and so BA is singular by Thm 1.

To show AB is singular, recall that rank A^T = rank A, so if rank $A < n$, then rank $A^T < n$ also. Then the cols. of A^T are also lin dep. so as above $\exists\, y \neq \underline{0}$ with

$$A^T y = \underline{0}.$$

Then $B^T A^T y = 0$

$$(B^T A^T) y = \underline{0}.$$

Thus the cols. of $B^T A^T$ are also lin. dep. again for the same reasons as before.

Thus $\operatorname{rank}(B^T A^T) < n$

But $\operatorname{rank}(B^T A^T) = \operatorname{rank}((AB)^T)$

$= \operatorname{rank}(AB)$

So $\operatorname{rank}(AB) < n$ and then AB is also singular, again by Thm I.